CHAPTER 16

THE FOURIER SERIES

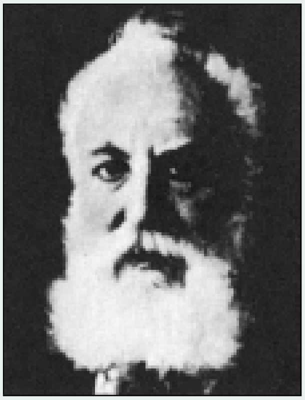
Do not worry about your difficulties in mathematics, I assure you that mine are greater.

—Albert Einstein

Historical Profiles

Jean Baptiste Joseph Fourier (1768–1830), a French mathematician, first presented the series and transform that bear his name. Fourier's results were not enthusiastically received by the scientific world. He could not even get his work published as a paper.

Born in Auxerre, France, Fourier was orphaned at age 8. He attended a local military college run by Benedictine monks, where he demonstrated great proficiency in mathematics. Like most of his contemporaries, Fourier was swept into the politics of the French Revolution. He played an important role in Napoleon's expeditions to Egypt in the later 1790s. Due to his political involvement, he narrowly escaped death twice.



Alexander Graham Bell (1847–1922) inventor of the telephone, was a Scottish-American scientist.

Bell was born in Edinburgh, Scotland, a son of Alexander Melville Bell, a well-known speech teacher. Alexander the younger also became a speech teacher after graduating from the University of Edinburgh and the University of London. In 1866 he became interested in transmitting speech electrically. After his older brother died of tuberculosis, his father decided to move to Canada. Alexander was asked to come to Boston to work at the School for the Deaf. There he met Thomas A. Watson, who became his assistant in his electromagnetic transmitter experiment. On March 10, 1876, Alexander sent the famous first telephone message: "Watson, come here I want you." The bel, the logarithmic unit introduced in Chapter 14, is named in his honor.

16.1 INTRODUCTION

We have spent a considerable amount of time on the analysis of circuits with sinusoidal sources. This chapter is concerned with a means of analyzing circuits with periodic, nonsinusoidal excitations. The notion of periodic functions was introduced in Chapter 9; it was mentioned there that the sinusoid is the most simple and useful periodic function. This chapter introduces the Fourier series, a technique for expressing a periodic function in terms of sinusoids. Once the source function is expressed in terms of sinusoids, we can apply the phasor method to analyze circuits.

The Fourier series is named after Jean Baptiste Joseph Fourier (1768–1830). In 1822, Fourier's genius came up with the insight that any practical periodic function can be represented as a sum of sinusoids. Such a representation, along with the superposition theorem, allows us to find the response of circuits to arbitrary periodic inputs using phasor techniques.

We begin with the trigonometric Fourier series. Later we consider the exponential Fourier series. We then apply Fourier series in circuit analysis. Finally, practical applications of Fourier series in spectrum analyzers and filters are demonstrated.

16.2 TRIGONOMETRIC FOURIER SERIES

While studying heat flow, Fourier discovered that a nonsinusoidal periodic function can be expressed as an infinite sum of sinusoidal functions. Recall that a periodic function is one that repeats every T seconds. In other words, a periodic function $f(t)$ satisfies

$$\boxed{f(t) = f(t + nT)} \tag{16.1}$$

where n is an integer and T is the period of the function.

According to the *Fourier theorem*, any practical periodic function of frequency ω_0 can be expressed as an infinite sum of sine or cosine functions that are integral multiples of ω_0. Thus, $f(t)$ can be expressed as

$$\begin{aligned} f(t) = a_0 &+ a_1 \cos \omega_0 t + b_1 \sin \omega_0 t + a_2 \cos 2\omega_0 t \\ &+ b_2 \sin 2\omega_0 t + a_3 \cos 3\omega_0 t + b_3 \sin 3\omega_0 t + \cdots \end{aligned} \tag{16.2}$$

or

$$\boxed{f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}}} \tag{16.3}$$

The harmonic frequency ω_n is an integral multiple of the fundamental frequency ω_0, i.e., $\omega_n = n\omega_0$.

where $\omega_0 = 2\pi/T$ is called the *fundamental frequency* in radians per second. The sinusoid $\sin n\omega_0 t$ or $\cos n\omega_0 t$ is called the nth harmonic of $f(t)$; it is an odd harmonic if n is odd and an even harmonic if n is even. Equation 16.3 is called the *trigonometric Fourier series* of $f(t)$. The constants a_n and b_n are the *Fourier coefficients*. The coefficient a_0 is the dc component or the average value of $f(t)$. (Recall that sinusoids

have zero average values.) The coefficients a_n and b_n (for $n \neq 0$) are the amplitudes of the sinusoids in the ac component. Thus,

> The **Fourier series** of a periodic function $f(t)$ is a representation that resolves $f(t)$ into a dc component and an ac component comprising an infinite series of harmonic sinusoids.

A function that can be represented by a Fourier series as in Eq. (16.3) must meet certain requirements, because the infinite series in Eq. (16.3) may or may not converge. These conditions on $f(t)$ to yield a convergent Fourier series are as follows:

1. $f(t)$ is single-valued everywhere.
2. $f(t)$ has a finite number of finite discontinuities in any one period.
3. $f(t)$ has a finite number of maxima and minima in any one period.
4. The integral $\int_{t_0}^{t_0+T} |f(t)|\,dt < \infty$ for any t_0.

These conditions are called *Dirichlet conditions*. Although they are not necessary conditions, they are sufficient conditions for a Fourier series to exist.

Historical note: Although Fourier published his theorem in 1822, it was P. G. L. Dirichlet (1805–1859) who later supplied an acceptable proof of the theorem.

A major task in Fourier series is the determination of the Fourier coefficients a_0, a_n, and b_n. The process of determining the coefficients is called *Fourier analysis*. The following trigonometric integrals are very helpful in Fourier analysis. For any integers m and n,

A software package like Mathcad or Maple can be used to evaluate the Fourier coefficients.

$$\int_0^T \sin n\omega_0 t\,dt = 0 \tag{16.4a}$$

$$\int_0^T \cos n\omega_0 t\,dt = 0 \tag{16.4b}$$

$$\int_0^T \sin n\omega_0 t \cos m\omega_0 t\,dt = 0 \tag{16.4c}$$

$$\int_0^T \sin n\omega_0 t \sin m\omega_0 t\,dt = 0, \qquad (m \neq n) \tag{16.4d}$$

$$\int_0^T \cos n\omega_0 t \cos m\omega_0 t\,dt = 0, \qquad (m \neq n) \tag{16.4e}$$

$$\int_0^T \sin^2 n\omega_0 t\,dt = \frac{T}{2} \tag{16.4f}$$

$$\int_0^T \cos^2 n\omega_0 t\,dt = \frac{T}{2} \tag{16.4g}$$

Let us use these identities to evaluate the Fourier coefficients.

We begin by finding a_0. We integrate both sides of Eq. (16.3) over one period and obtain

$$\int_0^T f(t)\,dt = \int_0^T \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\right] dt$$
$$= \int_0^T a_0\,dt + \sum_{n=1}^{\infty}\left[\int_0^T a_n \cos n\omega_0 t\,dt + \int_0^T b_n \sin n\omega_0 t\,dt\right] dt \tag{16.5}$$

Invoking the identities of Eqs. (16.4a) and (16.4b), the two integrals involving the ac terms vanish. Hence,

$$\int_0^T f(t)\,dt = \int_0^T a_0\,dt = a_0 T$$

or

$$\boxed{a_0 = \frac{1}{T}\int_0^T f(t)\,dt} \tag{16.6}$$

showing that a_0 is the average value of $f(t)$.

To evaluate a_n, we multiply both sides of Eq. (16.3) by $\cos m\omega_0 t$ and integrate over one period:

$$\int_0^T f(t)\cos m\omega_0 t\,dt$$
$$= \int_0^T \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\right]\cos m\omega_0 t\,dt$$
$$= \int_0^T a_0 \cos m\omega_0 t\,dt + \sum_{n=1}^{\infty}\left[\int_0^T a_n \cos n\omega_0 t \cos m\omega_0 t\,dt + \int_0^T b_n \sin n\omega_0 t \cos m\omega_0 t\,dt\right] dt \tag{16.7}$$

The integral containing a_0 is zero in view of Eq. (16.4b), while the integral containing b_n vanishes according to Eq. (16.4c). The integral containing a_n will be zero except when $m = n$, in which case it is $T/2$, according to Eqs. (16.4e) and (16.4g). Thus,

$$\int_0^T f(t)\cos m\omega_0 t\,dt = a_n \frac{T}{2}, \qquad \text{for } m = n$$

or

$$\boxed{a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt} \tag{16.8}$$

In a similar vein, we obtain b_n by multiplying both sides of Eq. (16.3) by $\sin m\omega_0 t$ and integrating over the period. The result is

$$\boxed{b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt} \tag{16.9}$$

Be aware that since $f(t)$ is periodic, it may be more convenient to carry the integrations above from $-T/2$ to $T/2$ or generally from t_0 to $t_0 + T$ instead of 0 to T. The result will be the same.

An alternative form of Eq. (16.3) is the *amplitude-phase* form

$$\boxed{f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)} \tag{16.10}$$

We can use Eqs. (9.11) and (9.12) to relate Eq. (16.3) to Eq. (16.10), or we can apply the trigonometric identity

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta \tag{16.11}$$

to the ac terms in Eq. (16.10) so that

$$\begin{aligned} a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n) = a_0 + \sum_{n=1}^{\infty} &(A_n \cos\phi_n) \cos n\omega_0 t \\ &- (A_n \sin\phi_n) \sin n\omega_0 t \end{aligned} \tag{16.12}$$

Equating the coefficients of the series expansions in Eqs. (16.3) and (16.12) shows that

$$a_n = A_n \cos\phi_n, \qquad b_n = -A_n \sin\phi_n \tag{16.13a}$$

or

$$\boxed{A_n = \sqrt{a_n^2 + b_n^2}, \qquad \phi_n = -\tan^{-1}\frac{b_n}{a_n}} \tag{16.13b}$$

To avoid any confusion in determining ϕ_n, it may be better to relate the terms in complex form as

$$A_n \underline{/\phi_n} = a_n - jb_n \tag{16.14}$$

The convenience of this relationship will become evident in Section 16.6. The plot of the amplitude A_n of the harmonics versus $n\omega_0$ is called the *amplitude spectrum* of $f(t)$; the plot of the phase ϕ_n versus $n\omega_0$ is the *phase spectrum* of $f(t)$. Both the amplitude and phase spectra form the *frequency spectrum* of $f(t)$.

The frequency spectrum is also known as the *line spectrum* in view of the discrete frequency components.

> The frequency spectrum of a signal consists of the plots of the amplitudes and phases of the harmonics versus frequency.

Thus, the Fourier analysis is also a mathematical tool for finding the spectrum of a periodic signal. Section 16.6 will elaborate more on the spectrum of a signal.

To evaluate the Fourier coefficients a_0, a_n, and b_n, we often need to apply the following integrals:

$$\int \cos at\, dt = \frac{1}{a}\sin at \tag{16.15a}$$

$$\int \sin at\, dt = -\frac{1}{a}\cos at \tag{16.15b}$$

$$\int t \cos at \, dt = \frac{1}{a^2} \cos at + \frac{1}{a} t \sin at \tag{16.15c}$$

$$\int t \sin at \, dt = \frac{1}{a^2} \sin at - \frac{1}{a} t \cos at \tag{16.15d}$$

It is also useful to know the values of the cosine, sine, and exponential functions for integral multiples of π. These are given in Table 16.1, where n is an integer.

TABLE 16.1 Values of cosine, sine, and exponential functions for integral multiples of π.

Function	Value
$\cos 2n\pi$	1
$\sin 2n\pi$	0
$\cos n\pi$	$(-1)^n$
$\sin n\pi$	0
$\cos \frac{n\pi}{2}$	$\begin{cases} (-1)^{n/2}, & n = \text{even} \\ 0, & n = \text{odd} \end{cases}$
$\sin \frac{n\pi}{2}$	$\begin{cases} (-1)^{(n-1)/2}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$
$e^{j2n\pi}$	1
$e^{jn\pi}$	$(-1)^n$
$e^{jn\pi/2}$	$\begin{cases} (-1)^{n/2}, & n = \text{even} \\ j(-1)^{(n-1)/2}, & n = \text{odd} \end{cases}$

EXAMPLE 16.1

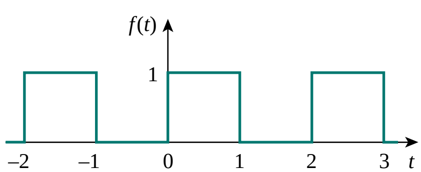


Figure 16.1 For Example 16.1; a square wave.

Determine the Fourier series of the waveform shown in Fig. 16.1. Obtain the amplitude and phase spectra.

Solution:

The Fourier series is given by Eq. (16.3), namely,

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \tag{16.1.1}$$

Our goal is to obtain the Fourier coefficients a_0, a_n, and b_n using Eqs. (16.6), (16.8), and (16.9). First, we describe the waveform as

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases} \tag{16.1.2}$$

and $f(t) = f(t + T)$. Since $T = 2$, $\omega_0 = 2\pi/T = \pi$. Thus,

$$a_0 = \frac{1}{T} \int_0^T f(t) \, dt = \frac{1}{2} \left[\int_0^1 1 \, dt + \int_1^2 0 \, dt \right] = \frac{1}{2} t \Big|_0^1 = \frac{1}{2} \tag{16.1.3}$$

Using Eq. (16.8) along with Eq. (16.15a),

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt \\ &= \frac{2}{2}\left[\int_0^1 1\cos n\pi t\,dt + \int_1^2 0\cos n\pi t\,dt\right] \\ &= \frac{1}{n\pi}\sin n\pi t\,\Big|_0^1 = \frac{1}{n\pi}\sin n\pi = 0 \end{aligned} \tag{16.1.4}$$

From Eq. (16.9) with the aid of Eq. (16.15b),

$$\begin{aligned} b_n &= \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt \\ &= \frac{2}{2}\left[\int_0^1 1\sin n\pi t\,dt + \int_1^2 0\sin n\pi t\,dt\right] \\ &= -\frac{1}{n\pi}\cos n\pi t\,\Big|_0^1 \\ &= -\frac{1}{n\pi}(\cos n\pi - 1), \qquad \cos n\pi = (-1)^n \\ &= \frac{1}{n\pi}[1-(-1)^n] = \begin{cases} \dfrac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases} \end{aligned} \tag{16.1.5}$$

Substituting the Fourier coefficients in Eqs. (16.1.3) to (16.1.5) into Eq. (16.1.1) gives the Fourier series as

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + \cdots \tag{16.1.6}$$

Since $f(t)$ contains only the dc component and the sine terms with the fundamental component and odd harmonics, it may be written as

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin n\pi t, \qquad n = 2k-1 \tag{16.1.7}$$

By summing the terms one by one as demonstrated in Fig. 16.2, we notice how superposition of the terms can evolve into the original square. As more and more Fourier components are added, the sum gets closer and closer to the square wave. However, it is not possible in practice to sum the series in Eq. (16.1.6) or (16.1.7) to infinity. Only a partial sum ($n = 1, 2, 3, \ldots, N$, where N is finite) is possible. If we plot the partial sum (or truncated series) over one period for a large N as in Fig. 16.3, we notice that the partial sum oscillates above and below the actual value of $f(t)$. At the neighborhood of the points of discontinuity ($x = 0, 1, 2, \ldots$), there is overshoot and damped oscillation. In fact, an overshoot of about 9 percent of the peak value is always present, regardless of the number of terms used to approximate $f(t)$. This is called the *Gibbs phenomenon*.

Summing the Fourier terms by hand calculation may be tedious. A computer is helpful to compute the terms and plot the sum like those shown in Fig. 16.2.

Historical note: Named after the mathematical physicist Josiah Willard Gibbs, who first observed it in 1899.

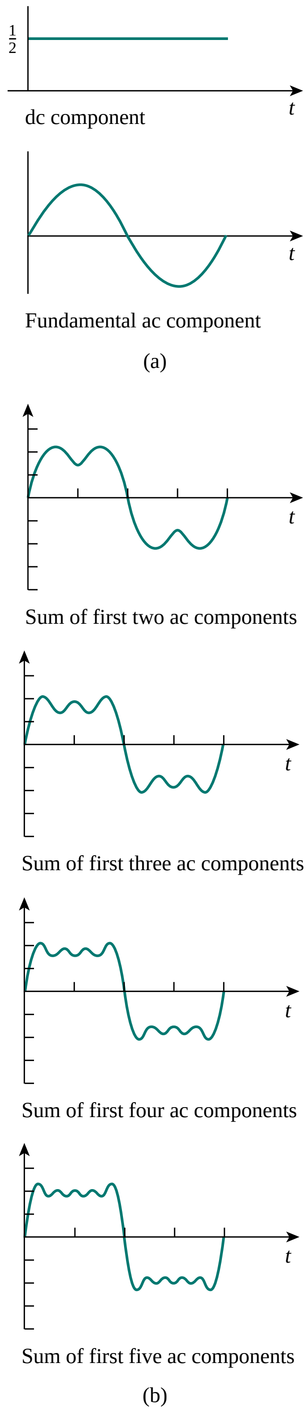


Figure 16.2 Evolution of a square wave from its Fourier components.

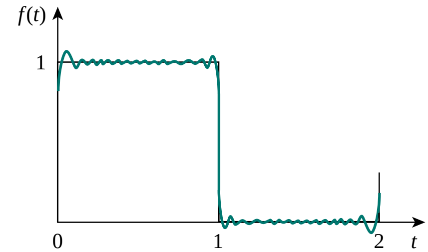


Figure 16.3 Truncating the Fourier series at $N = 11$; Gibbs phenomenon.

Finally, let us obtain the amplitude and phase spectra for the signal in Fig. 16.1. Since $a_n = 0$,

$$A_n = \sqrt{a_n^2 + b_n^2} = |b_n| = \begin{cases} \dfrac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases} \tag{16.1.8}$$

and

$$\phi_n = -\tan^{-1}\frac{b_n}{a_n} = \begin{cases} -90^\circ, & n = \text{odd} \\ 0, & n = \text{even} \end{cases} \tag{16.1.9}$$

The plots of A_n and ϕ_n for different values of $n\omega_0 = n\pi$ provide the amplitude and phase spectra in Fig. 16.4. Notice that the amplitudes of the harmonics decay very fast with frequency.

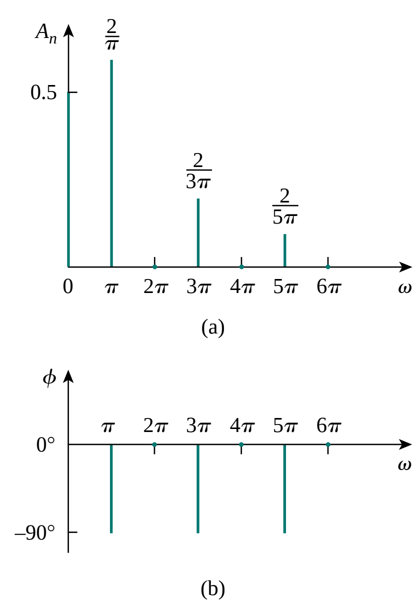


Figure 16.4 For Example 16.1: (a) amplitude and (b) phase spectrum of the function shown in Fig. 16.1.

PRACTICE PROBLEM 16.1

Find the Fourier series of the square wave in Fig. 16.5. Plot the amplitude and phase spectra.

Answer: $f(t) = \dfrac{4}{\pi}\displaystyle\sum_{k=1}^{\infty}\dfrac{1}{n}\sin n\pi t,\ n = 2k-1$. See Fig. 16.6 for the spectra.

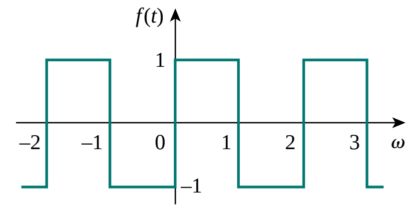


Figure 16.5 For Practice Prob. 16.1.

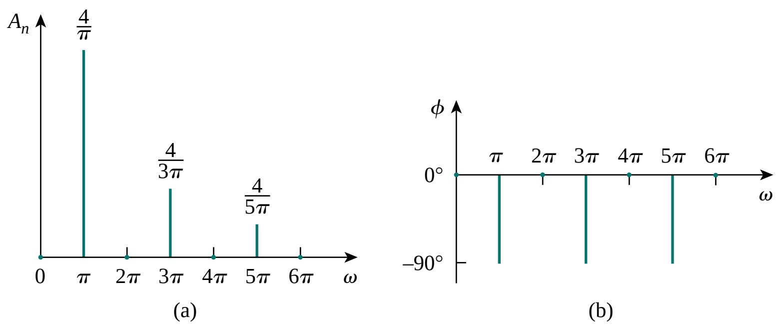


Figure 16.6 For Practice Prob. 16.1: amplitude and phase spectra for the function shown in Fig. 16.5.

EXAMPLE 16.2

Obtain the Fourier series for the periodic function in Fig. 16.7 and plot the amplitude and phase spectra.

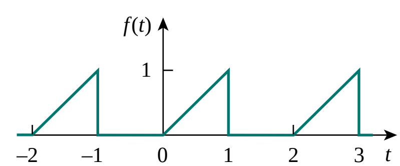


Figure 16.7 For Example 16.2.

Solution:

The function is described as

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

Since $T = 2$, $\omega_0 = 2\pi/T = \pi$. Then

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\left[\int_0^1 t\,dt + \int_1^2 0\,dt\right] = \frac{1}{2}\frac{t^2}{2}\bigg|_0^1 = \frac{1}{4} \tag{16.2.1}$$

To evaluate a_n and b_n, we need the integrals in Eq. (16.15):

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt \\ &= \frac{2}{2}\left[\int_0^1 t\cos n\pi t\,dt + \int_1^2 0\cos n\pi t\,dt\right] \\ &= \left[\frac{1}{n^2\pi^2}\cos n\pi t + \frac{t}{n\pi}\sin n\pi t\right]\bigg|_0^1 \\ &= \frac{1}{n^2\pi^2}(\cos n\pi - 1) + 0 = \frac{(-1)^n - 1}{n^2\pi^2} \end{aligned} \tag{16.2.2}$$

since $\cos n\pi = (-1)^n$; and

$$\begin{aligned} b_n &= \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt \\ &= \frac{2}{2}\left[\int_0^1 t\sin n\pi t\,dt + \int_1^2 0\sin n\pi t\,dt\right] \\ &= \left[\frac{1}{n^2\pi^2}\sin n\pi t - \frac{t}{n\pi}\cos n\pi t\right]\Bigg|_0^1 \\ &= 0 - \frac{\cos n\pi}{n\pi} = \frac{(-1)^{n+1}}{n\pi} \end{aligned} \tag{16.2.3}$$

Substituting the Fourier coefficients just found into Eq. (16.3) yields

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{[(-1)^n - 1]}{(n\pi)^2}\cos n\pi t + \frac{(-1)^{n+1}}{n\pi}\sin n\pi t\right]$$

To obtain the amplitude and phase spectra, we notice that, for even harmonics, $a_n = 0$, $b_n = -1/n\pi$, so that

$$A_n\underline{/\phi_n} = a_n - jb_n = 0 + j\frac{1}{n\pi} \tag{16.2.4}$$

Hence,

$$\begin{aligned} A_n &= |b_n| = \frac{1}{n\pi}, \qquad n = 2, 4, \ldots \\ \phi_n &= 90^\circ, \qquad n = 2, 4, \ldots \end{aligned} \tag{16.2.5}$$

For odd harmonics, $a_n = -2/(n^2\pi^2)$, $b_n = 1/(n\pi)$ so that

$$A_n\underline{/\phi_n} = a_n - jb_n = -\frac{2}{n^2\pi^2} - j\frac{1}{n\pi} \tag{16.2.6}$$

That is,

$$\begin{aligned} A_n &= \sqrt{a_n^2 + b_n^2} = \sqrt{\frac{4}{n^4\pi^4} + \frac{1}{n^2\pi^2}} \\ &= \frac{1}{n^2\pi^2}\sqrt{4 + n^2\pi^2}, \qquad n = 1, 3, \ldots \end{aligned} \tag{16.2.7}$$

From Eq. (16.2.6), we observe that ϕ lies in the third quadrant, so that

$$\phi_n = 180^\circ + \tan^{-1}\frac{n\pi}{2}, \qquad n = 1, 3, \ldots \tag{16.2.8}$$

From Eqs. (16.2.5), (16.2.7), and (16.2.8), we plot A_n and ϕ_n for different values of $n\omega_0 = n\pi$ to obtain the amplitude spectrum and phase spectrum as shown in Fig. 16.8.

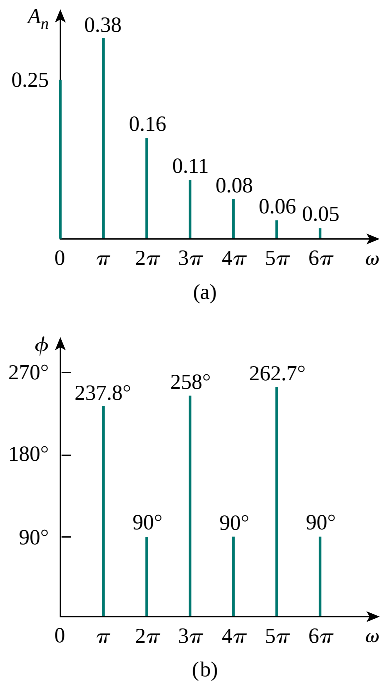


Figure 16.8 For Example 16.2: (a) amplitude spectrum, (b) phase spectrum.

PRACTICE PROBLEM 16.2

Determine the Fourier series of the sawtooth waveform in Fig. 16.9.

Answer: $f(t) = \dfrac{1}{2} - \dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n}\sin 2\pi n t.$

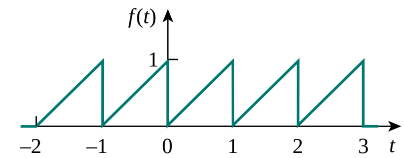


Figure 16.9 For Practice Prob. 16.2.

16.3 SYMMETRY CONSIDERATIONS

We noticed that the Fourier series of Example 16.1 consisted only of the sine terms. One may wonder if a method exists whereby one can know in advance that some Fourier coefficients would be zero and avoid the unnecessary work involved in the tedious process of calculating them. Such a method does exist; it is based on recognizing the existence of symmetry. Here we discuss three types of symmetry: (1) even symmetry, (2) odd symmetry, (3) half-wave symmetry.

16.3.1 Even Symmetry

A function $f(t)$ is *even* if its plot is symmetrical about the vertical axis; that is,

$$\boxed{f(t) = f(-t)} \tag{16.16}$$

Examples of even functions are t^2, t^4, and $\cos t$. Figure 16.10 shows more examples of periodic even functions. Note that each of these examples satisfies Eq. (16.16). A main property of an even function $f_e(t)$ is that:

$$\int_{-T/2}^{T/2} f_e(t)\,dt = 2\int_0^{T/2} f_e(t)\,dt \tag{16.17}$$

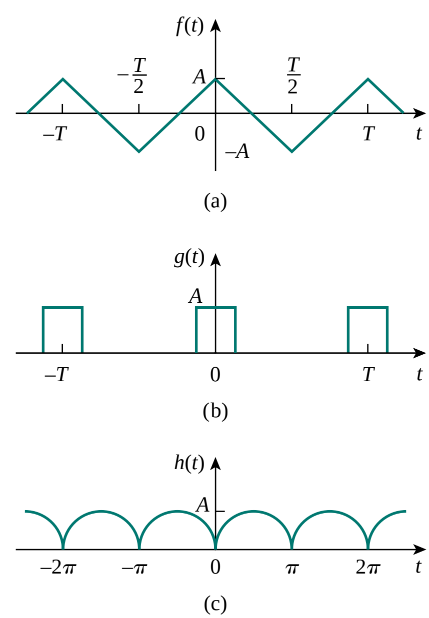


Figure 16.10 Typical examples of even periodic functions.

because integrating from $-T/2$ to 0 is the same as integrating from 0 to $T/2$. Utilizing this property, the Fourier coefficients for an even function become

$$\boxed{\begin{aligned} a_0 &= \frac{2}{T}\int_0^{T/2} f(t)\,dt \\ a_n &= \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt \\ b_n &= 0 \end{aligned}} \tag{16.18}$$

Since $b_n = 0$, Eq. (16.3) becomes a *Fourier cosine series*. This makes sense because the cosine function is itself even. It also makes intuitive sense that an even function contains no sine terms since the sine function is odd.

To confirm Eq. (16.18) quantitatively, we apply the property of an even function in Eq. (16.17) in evaluating the Fourier coefficients in Eqs. (16.6), (16.8), and (16.9). It is convenient in each case to integrate over the interval $-T/2 < t < T/2$, which is symmetrical about the origin. Thus,

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{T}\left[\int_{-T/2}^{0} f(t)\,dt + \int_{0}^{T/2} f(t)\,dt\right] \tag{16.19}$$

We change variables for the integral over the interval $-T/2 < t < 0$ by letting $t = -x$, so that $dt = -dx$, $f(t) = f(-t) = f(x)$, since $f(t)$ is an even function, and when $t = -T/2$, $x = T/2$. Then,

$$\begin{aligned} a_0 &= \frac{1}{T}\left[\int_{T/2}^{0} f(x)(-dx) + \int_{0}^{T/2} f(t)\,dt\right] \\ &= \frac{1}{T}\left[\int_{0}^{T/2} f(x)\,dx + \int_{0}^{T/2} f(t)\,dt\right] \end{aligned} \tag{16.20}$$

showing that the two integrals are identical. Hence,

$$a_0 = \frac{2}{T}\int_{0}^{T/2} f(t)\,dt \tag{16.21}$$

as expected. Similarly, from Eq. (16.8),

$$a_n = \frac{2}{T}\left[\int_{-T/2}^{0} f(t)\cos n\omega_0 t\,dt + \int_{0}^{T/2} f(t)\cos n\omega_0 t\,dt\right] \tag{16.22}$$

We make the same change of variables that led to Eq. (16.20) and note that both $f(t)$ and $\cos n\omega_0 t$ are even functions, implying that $f(-t) = f(t)$ and $\cos(-n\omega_0 t) = \cos n\omega_0 t$. Equation (16.22) becomes

$$\begin{aligned} a_n &= \frac{2}{T}\left[\int_{T/2}^{0} f(-x)\cos(-n\omega_0 x)(-dx) + \int_{0}^{T/2} f(t)\cos n\omega_0 t\,dt\right] \\ &= \frac{2}{T}\left[\int_{T/2}^{0} f(x)\cos(n\omega_0 x)(-dx) + \int_{0}^{T/2} f(t)\cos n\omega_0 t\,dt\right] \\ &= \frac{2}{T}\left[\int_{0}^{T/2} f(x)\cos(n\omega_0 x)\,dx + \int_{0}^{T/2} f(t)\cos n\omega_0 t\,dt\right] \end{aligned} \tag{16.23a}$$

or

$$a_n = \frac{4}{T}\int_{0}^{T/2} f(t)\cos n\omega_0 t\,dt \tag{16.23b}$$

as expected. For b_n, we apply Eq. (16.9),

$$b_n = \frac{2}{T}\left[\int_{-T/2}^{0} f(t)\sin n\omega_0 t\,dt + \int_{0}^{T/2} f(t)\sin n\omega_0 t\,dt\right] \tag{16.24}$$

We make the same change of variables but keep in mind that $f(-t) = f(t)$ but $\sin(-n\omega_0 t) = -\sin n\omega_0 t$. Equation (16.24) yields

$$\begin{aligned} b_n &= \frac{2}{T}\left[\int_{T/2}^{0} f(-x)\sin(-n\omega_0 x)(-dx) + \int_{0}^{T/2} f(t)\sin n\omega_0 t\,dt\right] \\ &= \frac{2}{T}\left[\int_{T/2}^{0} f(x)\sin n\omega_0 x\,dx + \int_{0}^{T/2} f(t)\sin n\omega_0 t\,dt\right] \\ &= \frac{2}{T}\left[-\int_{0}^{T/2} f(x)\sin(n\omega_0 x)\,dx + \int_{0}^{T/2} f(t)\sin n\omega_0 t\,dt\right] \\ &= 0 \end{aligned} \tag{16.25}$$

confirming Eq. (16.18).

16.3.2 Odd Symmetry

A function $f(t)$ is said to be *odd* if its plot is antisymmetrical about the vertical axis:

$$\boxed{f(-t) = -f(t)} \tag{16.26}$$

Examples of odd functions are t, t^3, and $\sin t$. Figure 16.11 shows more examples of periodic odd functions. All these examples satisfy Eq. (16.26). An odd function $f_o(t)$ has this major characteristic:

$$\int_{-T/2}^{T/2} f_o(t)\,dt = 0 \tag{16.27}$$

because integration from $-T/2$ to 0 is the negative of that from 0 to $T/2$. With this property, the Fourier coefficients for an odd function become

$$\boxed{\begin{gathered} a_0 = 0, \qquad a_n = 0 \\ b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\,dt \end{gathered}} \tag{16.28}$$

which give us a *Fourier sine series*. Again, this makes sense because the sine function is itself an odd function. Also, note that there is no dc term for the Fourier series expansion of an odd function.

The quantitative proof of Eq. (16.28) follows the same procedure taken to prove Eq. (16.18) except that $f(t)$ is now odd, so that $f(t) = -f(t)$. With this fundamental but simple difference, it is easy to see that $a_0 = 0$ in Eq. (16.20), $a_n = 0$ in Eq. (16.23a), and b_n in Eq. (16.24) becomes

$$\begin{aligned} b_n &= \frac{2}{T}\left[\int_{T/2}^{0} f(-x)\sin(-n\omega_0 x)(-dx) + \int_0^{T/2} f(t)\sin n\omega_0 t\,dt\right] \\ &= \frac{2}{T}\left[-\int_{T/2}^{0} f(x)\sin n\omega_0 x\,dx + \int_0^{T/2} f(t)\sin n\omega_0 t\,dt\right] \\ &= \frac{2}{T}\left[\int_0^{T/2} f(x)\sin(n\omega_0 x)\,dx + \int_0^{T/2} f(t)\sin n\omega_0 t\,dt\right] \end{aligned}$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\,dt \tag{16.29}$$

as expected.

It is interesting to note that any periodic function $f(t)$ with neither even nor odd symmetry may be decomposed into even and odd parts. Using the properties of even and odd functions from Eqs. (16.16) and (16.26), we can write

$$f(t) = \underbrace{\frac{1}{2}[f(t) + f(-t)]}_{\text{even}} + \underbrace{\frac{1}{2}[f(t) - f(-t)]}_{\text{odd}} = f_e(t) + f_o(t) \tag{16.30}$$

Notice that $f_e(t) = \frac{1}{2}[f(t) + f(-t)]$ satisfies the property of an even function in Eq. (16.16), while $f_o(t) = \frac{1}{2}[f(t) - f(-t)]$ satisfies the property of an odd function in Eq. (16.26). The fact that $f_e(t)$ contains

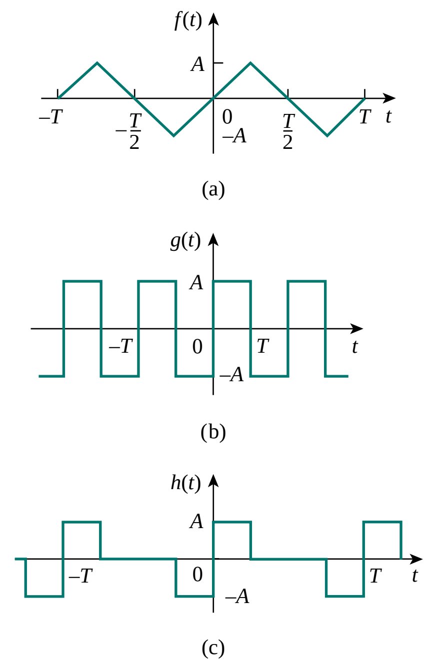


Figure 16.11 Typical examples of odd periodic functions.

only the dc term and the cosine terms, while $f_o(t)$ has only the sine terms, can be exploited in grouping the Fourier series expansion of $f(t)$ as

$$f(t) = \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t}_{\text{even}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin n\omega_0 t}_{\text{odd}} = f_e(t) + f_o(t) \quad (16.31)$$

It follows readily from Eq. (16.31) that when $f(t)$ is even, $b_n = 0$, and when $f(t)$ is odd, $a_0 = 0 = a_n$.

Also, note the following properties of odd and even functions:

1. The product of two even functions is also an even function.
2. The product of two odd functions is an even function.
3. The product of an even function and an odd function is an odd function.
4. The sum (or difference) of two even functions is also an even function.
5. The sum (or difference) of two odd functions is an odd function.
6. The sum (or difference) of an even function and an odd function is neither even nor odd.

Each of these properties can be proved using Eqs. (16.16) and (16.26).

16.3.3 Half-Wave Symmetry

A function is half-wave (odd) symmetric if

$$\boxed{f\left(t - \frac{T}{2}\right) = -f(t)} \quad (16.32)$$

which means that each half-cycle is the mirror image of the next half-cycle. Notice that functions $\cos n\omega_0 t$ and $\sin n\omega_0 t$ satisfy Eq. (16.32) for odd values of n and therefore possess half-wave symmetry when n is odd. Figure 16.12 shows other examples of half-wave symmetric functions. The functions in Figs. 16.11(a) and 16.11(b) are also half-wave symmetric. Notice that for each function, one half-cycle is the inverted

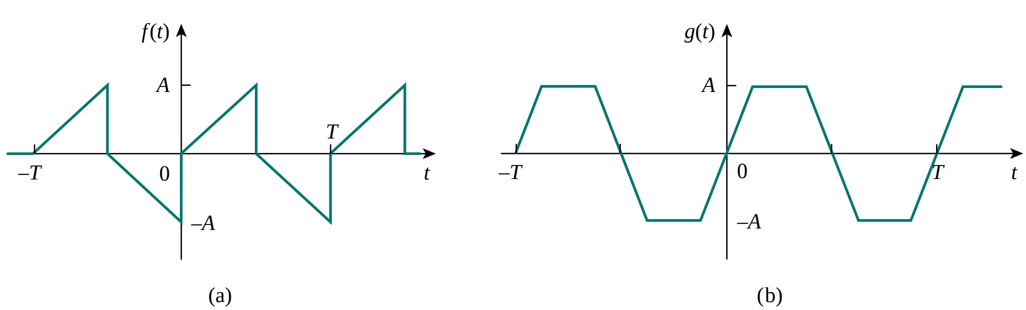


Figure 16.12 Typical examples of half-wave odd symmetric functions.

version of the adjacent half-cycle. The Fourier coefficients become

$$\boxed{\begin{aligned} a_0 &= 0 \\ a_n &= \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\cos n\omega_0 t\,dt, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases} \\ b_n &= \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\sin n\omega_0 t\,dt, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases} \end{aligned}} \tag{16.33}$$

showing that the Fourier series of a half-wave symmetric function contains only odd harmonics.

To derive Eq. (16.33), we apply the property of half-wave symmetric functions in Eq. (16.32) in evaluating the Fourier coefficients in Eqs. (16.6), (16.8), and (16.9). Thus,

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{T}\left[\int_{-T/2}^{0} f(t)\,dt + \int_0^{T/2} f(t)\,dt\right] \tag{16.34}$$

We change variables for the integral over the interval $-T/2 < t < 0$ by letting $x = t + T/2$, so that $dx = dt$; when $t = -T/2, x = 0$; and when $t = 0, x = T/2$. Also, we keep Eq. (16.32) in mind; that is, $f(x - T/2) = -f(x)$. Then,

$$\begin{aligned} a_0 &= \frac{1}{T}\left[\int_0^{T/2} f\left(x - \frac{T}{2}\right)dx + \int_0^{T/2} f(t)\,dt\right] \\ &= \frac{1}{T}\left[-\int_0^{T/2} f(x)\,dx + \int_0^{T/2} f(t)\,dt\right] = 0 \end{aligned} \tag{16.35}$$

confirming the expression for a_0 in Eq. (16.33). Similarly,

$$a_n = \frac{2}{T}\left[\int_{-T/2}^{0} f(t)\cos n\omega_0 t\,dt + \int_0^{T/2} f(t)\cos n\omega_0 t\,dt\right] \tag{16.36}$$

We make the same change of variables that led to Eq. (16.35) so that Eq. (16.36) becomes

$$\begin{aligned} a_n = \frac{2}{T}\Bigg[&\int_0^{T/2} f\left(x - \frac{T}{2}\right)\cos n\omega_0\left(x - \frac{T}{2}\right)dx \\ &+ \int_0^{T/2} f(t)\cos n\omega_0 t\,dt\Bigg] \end{aligned} \tag{16.37}$$

Since $f(x - T/2) = -f(x)$ and

$$\begin{aligned} \cos n\omega_0\left(x - \frac{T}{2}\right) &= \cos(n\omega_0 t - n\pi) \\ &= \cos n\omega_0 t\cos n\pi + \sin n\omega_0 t\sin n\pi \\ &= (-1)^n\cos n\omega_0 t \end{aligned} \tag{16.38}$$

substituting these in Eq. (16.37) leads to

$$
\begin{aligned}
a_n &= \frac{2}{T}[1-(-1)^n]\int_0^{T/2} f(t)\cos n\omega_0 t\,dt \\
&= \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\cos n\omega_0 t\,dt, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases}
\end{aligned}
\tag{16.39}
$$

confirming Eq. (16.33). By following a similar procedure, we can derive b_n as in Eq. (16.33).

Table 16.2 summarizes the effects of these symmetries on the Fourier coefficients. Table 16.3 provides the Fourier series of some common periodic functions.

TABLE 16.2 Effects of symmetry on Fourier coefficients.

Symmetry	a_0	a_n	b_n	Remarks
Even	$a_0 \neq 0$	$a_n \neq 0$	$b_n = 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.
Odd	$a_0 = 0$	$a_n = 0$	$b_n \neq 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.
Half-wave	$a_0 = 0$	$a_{2n} = 0$ $a_{2n+1} \neq 0$	$b_{2n} = 0$ $b_{2n+1} \neq 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.

TABLE 16.3 The Fourier series of common functions.

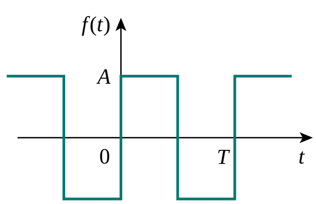
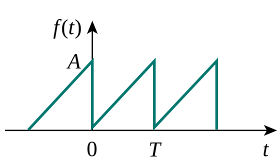
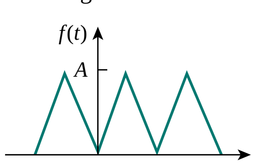
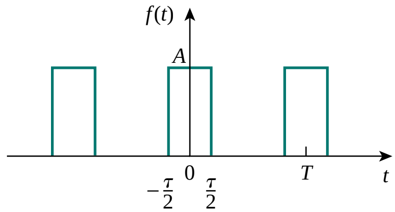
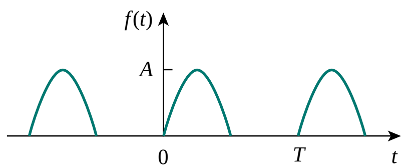
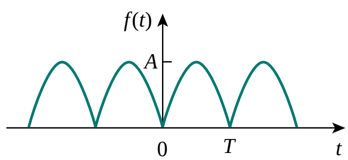
Function	Fourier series
1. Square wave 	$f(t) = \dfrac{4A}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{2n-1}\sin(2n-1)\omega_0 t$
2. Sawtooth wave 	$f(t) = \dfrac{A}{2} - \dfrac{A}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{\sin n\omega_0 t}{n}$
3. Triangular wave 	$f(t) = \dfrac{A}{2} - \dfrac{4A}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{(2n+1)^2}\cos(2n-1)\omega_0 t$

TABLE 16.3 *(continued)*

Function	Fourier series
4. Rectangular pulse train 	$f(t) = \frac{A\tau}{T} + \frac{2A}{T}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi\tau}{T}\cos n\omega_0 t$
5. Half-wave rectified sine 	$f(t) = \frac{A}{\pi} + \frac{A}{2}\sin\omega_0 t - \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2n\omega_0 t$
6. Full-wave rectified sine 	$f(t) = \frac{2A}{\pi} - \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos n\omega_0 t$

EXAMPLE 16.3

Find the Fourier series expansion of $f(t)$ given in Fig. 16.13.

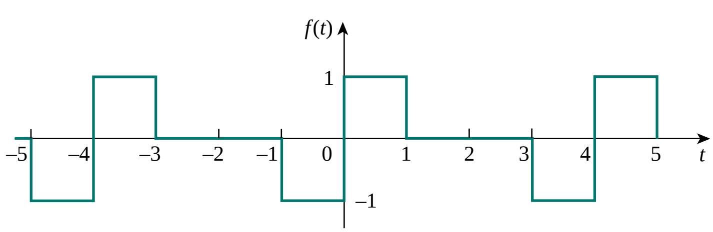


Figure 16.13 For Example 16.3.

Solution:

The function $f(t)$ is an odd function. Hence $a_0 = 0 = a_n$. The period is $T = 4$, and $\omega_0 = 2\pi/T = \pi/2$, so that

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\, dt$$
$$= \frac{4}{4}\left[\int_0^1 1\sin\frac{n\pi}{2}t\, dt + \int_1^2 0\sin\frac{n\pi}{2}t\, dt\right]$$
$$= -\frac{2}{n\pi}\cos\frac{n\pi t}{2}\bigg|_0^1 = \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$

Hence,

$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}t$$

which is a Fourier sine series.

PRACTICE PROBLEM 16.3

Find the Fourier series of the function $f(t)$ in Fig. 16.14.

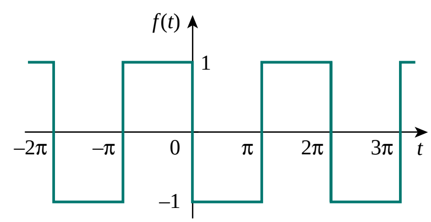


Figure 16.14 For Practice Prob. 16.3.

Answer: $f(t) = -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin nt,\ n = 2k-1.$

EXAMPLE 16.4

Determine the Fourier series for the half-wave rectified cosine function shown in Fig. 16.15.

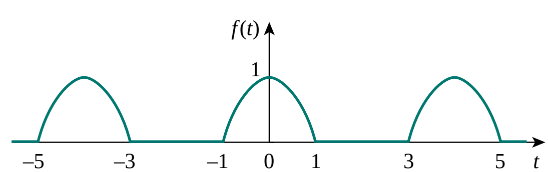


Figure 16.15 A half-wave rectified cosine function; for Example 16.4.

Solution:

This is an even function so that $b_n = 0$. Also, $T = 4$, $\omega_0 = 2\pi/T = \pi/2$. Over a period,

$$f(t) = \begin{cases} 0, & -2 < t < -1 \\ \cos \dfrac{\pi}{2} t, & -1 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt = \frac{2}{4}\left[\int_0^1 \cos\frac{\pi}{2}t\,dt + \int_1^2 0\,dt\right]$$

$$= \frac{1}{2}\frac{2}{\pi}\sin\frac{\pi}{2}t\Big|_0^1 = \frac{1}{\pi}$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt = \frac{4}{4}\left[\int_0^1 \cos\frac{\pi}{2}t\cos\frac{n\pi t}{2}\,dt + 0\right]$$

But $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$. Then

$$a_n = \frac{1}{2}\int_0^1 \left[\cos\frac{\pi}{2}(n+1)t + \cos\frac{\pi}{2}(n-1)t\right] dt$$

For $n = 1$,

$$a_1 = \frac{1}{2}\int_0^1 [\cos\pi t + 1]\,dt = \frac{1}{2}\left[\frac{\sin\pi t}{\pi} + t\right]\Big|_0^1 = \frac{1}{2}$$

For $n > 1$,

$$a_n = \frac{1}{\pi(n+1)}\sin\frac{\pi}{2}(n+1) + \frac{1}{\pi(n-1)}\sin\frac{\pi}{2}(n-1)$$

For $n =$ odd ($n = 1, 3, 5, \ldots$), $(n+1)$ and $(n-1)$ are both even, so

$$\sin\frac{\pi}{2}(n+1) = 0 = \sin\frac{\pi}{2}(n-1), \qquad n = \text{odd}$$

For $n =$ even ($n = 2, 4, 6, \ldots$), $(n+1)$ and $(n-1)$ are both odd. Also,

$$\sin\frac{\pi}{2}(n+1) = -\sin\frac{\pi}{2}(n-1) = \cos\frac{n\pi}{2} = (-1)^{n/2}, \qquad n = \text{even}$$

Hence,

$$a_n = \frac{(-1)^{n/2}}{\pi(n+1)} + \frac{-(-1)^{n/2}}{\pi(n-1)} = \frac{-2(-1)^{n/2}}{\pi(n^2-1)}, \qquad n = \text{even}$$

Thus,

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\cos\frac{\pi}{2}t - \frac{2}{\pi}\sum_{n=\text{even}}^{\infty}\frac{(-1)^{n/2}}{(n^2-1)}\cos\frac{n\pi}{2}t$$

To avoid using $n = 2, 4, 6, \ldots$ and also to ease computation, we can replace n by $2k$, where $k = 1, 2, 3, \ldots$ and obtain

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\cos\frac{\pi}{2}t - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(4k^2-1)}\cos k\pi t$$

which is a Fourier cosine series.

PRACTICE PROBLEM 16.4

Find the Fourier series expansion of the function in Fig. 16.16.

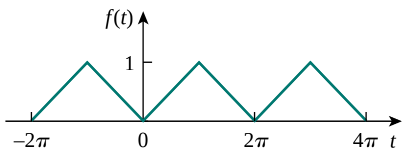


Figure 16.16 For Practice Prob. 16.4.

Answer: $f(t) = \dfrac{1}{2} - \dfrac{4}{\pi^2}\sum_{k=1}^{\infty}\dfrac{1}{n^2}\cos nt,\ n = 2k-1.$

EXAMPLE 16.5

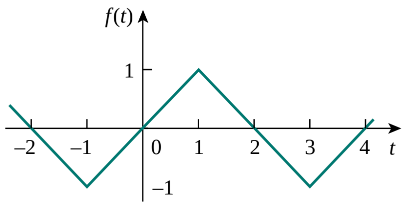


Figure 16.17 For Example 16.5.

Calculate the Fourier series for the function in Fig. 16.17.

Solution:

The function in Fig. 16.17 is half-wave odd symmetric, so that $a_0 = 0 = a_n$. It is described over half the period as

$$f(t) = t, \qquad -1 < t < 1$$

$T = 4$, $\omega_0 = 2\pi/T = \pi/2$. Hence,

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\, dt$$

Instead of integrating $f(t)$ from 0 to 2, it is more convenient to integrate from -1 to 1. Applying Eq. (16.15d),

$$\begin{aligned}
b_n &= \frac{4}{4}\int_{-1}^{1} t\sin\frac{n\pi t}{2}\,dt = \left[\frac{\sin n\pi t/2}{n^2\pi^2/4} - \frac{t\cos n\pi t/2}{n\pi/2}\right]\Bigg|_{-1}^{1} \\
&= \frac{4}{n^2\pi^2}\left[\sin\frac{n\pi}{2} - \sin\left(-\frac{n\pi}{2}\right)\right] - \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} + \cos\left(-\frac{n\pi}{2}\right)\right] \\
&= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{4}{n\pi}\cos\frac{n\pi}{2}
\end{aligned}$$

since $\sin(-x) = -\sin x$ as an odd function, while $\cos(-x) = \cos x$ as an even function. Using the identities for $\sin n\pi/2$ and $\cos n\pi/2$ in Table 16.1,

$$b_n = \begin{cases} \dfrac{8}{n^2\pi^2}(-1)^{(n-1)/2}, & n = \text{odd} = 1, 3, 5, \ldots \\ \dfrac{4}{n\pi}(-1)^{(n+2)/2}, & n = \text{even} = 2, 4, 6, \ldots \end{cases}$$

Thus,

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{2}t$$

where b_n is given above.

PRACTICE PROBLEM 16.5

Determine the Fourier series of the function in Fig. 16.12(a). Take $A = 1$ and $T = 2\pi$.

Answer: $f(t) = \dfrac{2}{\pi}\sum_{k=1}^{\infty}\left(\dfrac{-2}{n^2\pi}\cos nt + \dfrac{1}{n}\sin nt\right),\ n = 2k-1.$

16.4 CIRCUIT APPLICATIONS

We find that in practice, many circuits are driven by nonsinusoidal periodic functions. To find the steady-state response of a circuit to a nonsinusoidal periodic excitation requires the application of a Fourier series, ac phasor analysis, and the superposition principle. The procedure usually involves three steps.

Steps for Applying Fourier Series:

1. Express the excitation as a Fourier series.
2. Find the response of each term in the Fourier series.
3. Add the individual responses using the superposition principle.

The first step is to determine the Fourier series expansion of the excitation. For the periodic voltage source shown in Fig. 16.18(a), for example, the Fourier series is expressed as

$$v(t) = V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t + \theta_n) \tag{16.40}$$

(The same could be done for a periodic current source.) Equation (16.40) shows that $v(t)$ consists of two parts: the dc component V_0 and the ac component $\mathbf{V}_n = V_n \angle \theta_n$ with several harmonics. This Fourier series representation may be regarded as a set of series-connected sinusoidal sources, with each source having its own amplitude and frequency, as shown in Fig. 16.18(b).

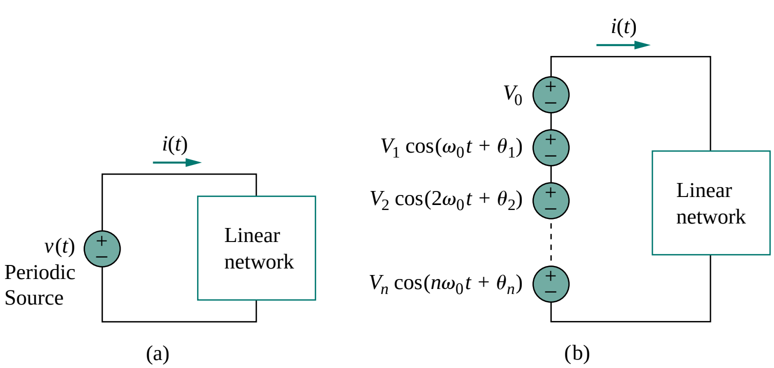


Figure 16.18 (a) Linear network excited by a periodic voltage source, (b) Fourier series representation (time-domain).

The second step is finding the response to each term in the Fourier series. The response to the dc component can be determined in the frequency domain by setting $n = 0$ or $\omega = 0$ as in Fig. 16.19(a), or in the time domain by replacing all inductors with short circuits and all capacitors with open circuits. The response to the ac component is obtained by the phasor techniques covered in Chapter 9, as shown in Fig. 16.19(b). The network is represented by its impedance $\mathbf{Z}(n\omega_0)$ or admittance $\mathbf{Y}(n\omega_0)$. $\mathbf{Z}(n\omega_0)$ is the input impedance at the source when ω is everywhere replaced by $n\omega_0$, and $\mathbf{Y}(n\omega_0)$ is the reciprocal of $\mathbf{Z}(n\omega_0)$.

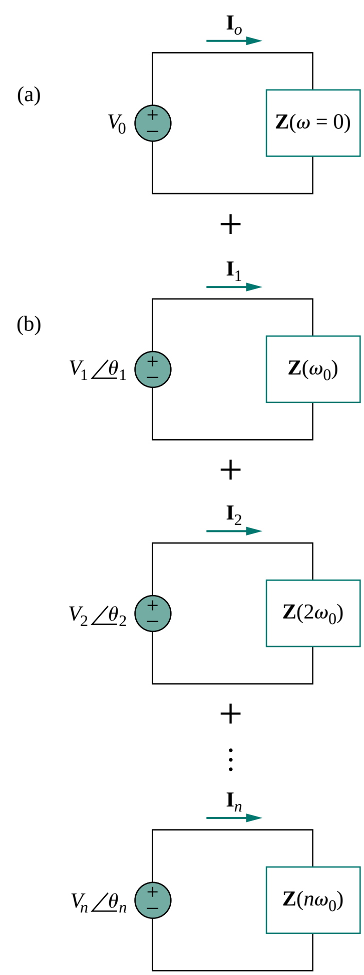


Figure 16.19 Steady-state responses: (a) dc component, (b) ac component (frequency domain).

Finally, following the principle of superposition, we add all the individual responses. For the case shown in Fig. 16.19,

$$\begin{aligned} i(t) &= i_0(t) + i_1(t) + i_2(t) + \cdots \\ &= \mathbf{I}_0 + \sum_{n=1}^{\infty} |\mathbf{I}_n| \cos(n\omega_0 t + \psi_n) \end{aligned} \tag{16.41}$$

where each component $\mathbf{I}_n$ with frequency $n\omega_0$ has been transformed to the time domain to get $i_n(t)$, and ψ_n is the argument of $\mathbf{I}_n$.

EXAMPLE 16.6

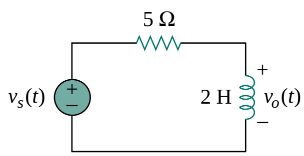


Figure 16.20 For Example 16.6.

Let the function $f(t)$ in Example 16.1 be the voltage source $v_s(t)$ in the circuit of Fig. 16.20. Find the response $v_o(t)$ of the circuit.

Solution:

From Example 16.1,

$$v_s(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin n\pi t, \qquad n = 2k-1$$

where $\omega_n = n\omega_0 = n\pi$ rad/s. Using phasors, we obtain the response $\mathbf{V}_o$ in the circuit of Fig. 16.20 by voltage division:

$$\mathbf{V}_o = \frac{j\omega_n L}{R + j\omega_n L}\mathbf{V}_s = \frac{j2n\pi}{5 + j2n\pi}\mathbf{V}_s$$

For the dc component ($\omega_n = 0$ or $n = 0$)

$$\mathbf{V}_s = \frac{1}{2} \qquad \Longrightarrow \qquad \mathbf{V}_o = 0$$

This is expected, since the inductor is a short circuit to dc. For the nth harmonic,

$$\mathbf{V}_s = \frac{2}{n\pi}\underline{/-90^\circ} \tag{16.6.1}$$

and the corresponding response is

$$\begin{aligned} \mathbf{V}_o &= \frac{2n\pi\underline{/90^\circ}}{\sqrt{25 + 4n^2\pi^2}\,\underline{/\tan^{-1} 2n\pi/5}}\,\frac{2}{n\pi}\underline{/-90^\circ} \\ &= \frac{4\underline{/-\tan^{-1}\ 2n\pi/5}}{\sqrt{25 + 4n^2\pi^2}} \end{aligned} \tag{16.6.2}$$

In the time domain,

$$v_o(t) = \sum_{k=1}^{\infty}\frac{4}{\sqrt{25 + 4n^2\pi^2}}\cos\left(n\pi t - \tan^{-1}\frac{2n\pi}{5}\right), \qquad n = 2k-1$$

The first three terms ($k = 1, 2, 3$ or $n = 1, 3, 5$) of the odd harmonics in the summation give us

$$\begin{aligned} v_o(t) = {} & 0.4981\cos(\pi t - 51.49^\circ) + 0.2051\cos(3\pi t - 75.14^\circ) \\ & + 0.1257\cos(5\pi t - 80.96^\circ) + \cdots \text{ V} \end{aligned}$$

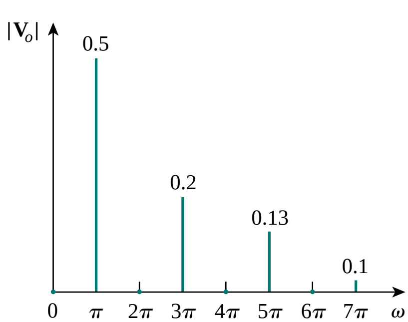


Figure 16.21 For Example 16.6: Amplitude spectrum of the output voltage.

Figure 16.21 shows the amplitude spectrum for output voltage $v_o(t)$, while that of the input voltage $v_s(t)$ is in Fig. 16.4(a). Notice that the

two spectra are close. Why? We observe that the circuit in Fig. 16.20 is a highpass filter with the corner frequency $\omega_c = R/L = 2.5$ rad/s, which is less than the fundamental frequency $\omega_0 = \pi$ rad/s. The dc component is not passed and the first harmonic is slightly attenuated, but higher harmonics are passed. In fact, from Eqs. (16.6.1) and (16.6.2), $\mathbf{V}_o$ is identical to $\mathbf{V}_s$ for large n, which is characteristic of a highpass filter.

PRACTICE PROBLEM 16.6

If the sawtooth waveform in Fig. 16.9 (see Practice Prob. 16.2) is the voltage source $v_s(t)$ in the circuit of Fig. 16.22, find the response $v_o(t)$.

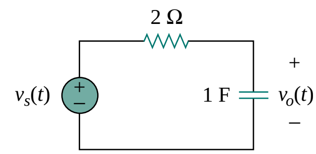


Figure 16.22 For Practice Prob. 16.6.

Answer: $v_o(t) = \dfrac{1}{2} - \dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{\sin(2\pi nt - \tan^{-1} 4n\pi)}{n\sqrt{1+16n^2\pi^2}}$ V.

EXAMPLE 16.7

Find the response $i_o(t)$ in the circuit in Fig. 16.23 if the input voltage $v(t)$ has the Fourier series expansion

$$v(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2}(\cos nt - n\sin nt)$$

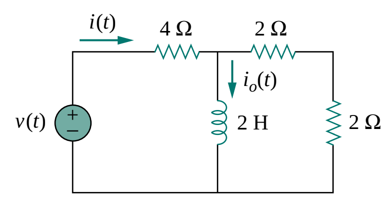


Figure 16.23 For Example 16.7.

Solution:

Using Eq. (16.13), we can express the input voltage as

$$\begin{aligned} v(t) &= 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\sqrt{1+n^2}}\cos(nt + \tan^{-1} n) \\ &= 1 - 1.414\cos(t + 45^\circ) + 0.8944\cos(2t + 63.45^\circ) \\ &\quad - 0.6345\cos(3t + 71.56^\circ) - 0.4851\cos(4t + 78.7^\circ) + \cdots \end{aligned}$$

We notice that $\omega_0 = 1$, $\omega_n = n$ rad/s. The impedance at the source is

$$\mathbf{Z} = 4 + j\omega_n 2 \parallel 4 = 4 + \frac{j\omega_n 8}{4 + j\omega_n 2} = \frac{8 + j\omega_n 8}{2 + j\omega_n}$$

The input current is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{2 + j\omega_n}{8 + j\omega_n 8}\mathbf{V}$$

where $\mathbf{V}$ is the phasor form of the source voltage $v(t)$. By current division,

$$\mathbf{I}_o = \frac{4}{4 + j\omega_n 2}\mathbf{I} = \frac{\mathbf{V}}{4 + j\omega_n 4}$$

Since $\omega_n = n$, $\mathbf{I}_o$ can be expressed as

$$\mathbf{I}_o = \frac{\mathbf{V}}{4\sqrt{1+n^2}\,\underline{/\tan^{-1} n}}$$

For the dc component ($\omega_n = 0$ or $n = 0$)

$$\mathbf{V} = 1 \qquad \Longrightarrow \qquad \mathbf{I}_o = \frac{\mathbf{V}}{4} = \frac{1}{4}$$

For the nth harmonic,

$$\mathbf{V} = \frac{2(-1)^n}{\sqrt{1+n^2}} \underline{/\tan^{-1} n}$$

so that

$$\mathbf{I}_o = \frac{1}{4\sqrt{1+n^2} \underline{/\tan^{-1} n}} \frac{2(-1)^n}{\sqrt{1+n^2}} \underline{/\tan^{-1} n} = \frac{(-1)^n}{2(1+n^2)}$$

In the time domain,

$$i_o(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2(1+n^2)} \cos nt \text{ A}$$

PRACTICE PROBLEM 16.7

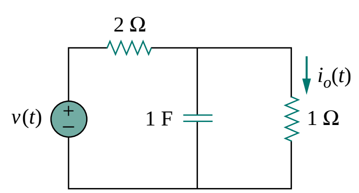


Figure 16.24 For Practice Prob. 16.7.

If the input voltage in the circuit of Fig. 16.24 is

$$v(t) = \frac{1}{3} + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \cos nt - \frac{\pi}{n} \sin nt \right) \text{ V}$$

determine the response $i_o(t)$.

Answer: $\frac{1}{9} + \sum_{n=1}^{\infty} \frac{\sqrt{1+n^2\pi^2}}{n^2\pi^2\sqrt{9+4n^2}} \cos\left(nt - \tan^{-1}\frac{2n}{3} + \tan^{-1} n\pi\right)$ A.

16.5 AVERAGE POWER AND RMS VALUES

Recall the concepts of average power and rms value of a periodic signal that we discussed in Chapter 11. To find the average power absorbed by a circuit due to a periodic excitation, we write the voltage and current in amplitude-phase form [see Eq. (16.10)] as

$$v(t) = V_{\text{dc}} + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t - \theta_n) \tag{16.42}$$

$$i(t) = I_{\text{dc}} + \sum_{m=1}^{\infty} I_m \cos(m\omega_0 t - \phi_m) \tag{16.43}$$

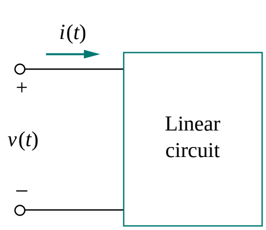


Figure 16.25 The voltage polarity reference and current reference direction.

Following the passive sign convention (Fig. 16.25), the average power is

$$P = \frac{1}{T} \int_0^T vi\, dt \tag{16.44}$$

Substituting Eqs. (16.42) and (16.43) into Eq. (16.44) gives

$$\begin{aligned} P = {} & \frac{1}{T} \int_0^T V_{\text{dc}} I_{\text{dc}}\, dt + \sum_{m=1}^{\infty} \frac{I_m V_{\text{dc}}}{T} \int_0^T \cos(m\omega_0 t - \phi_m)\, dt \\ & + \sum_{n=1}^{\infty} \frac{V_n I_{\text{dc}}}{T} \int_0^T \cos(n\omega_0 t - \theta_n)\, dt \\ & + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{V_n I_m}{T} \int_0^T \cos(n\omega_0 t - \theta_n) \cos(m\omega_0 t - \phi_m)\, dt \end{aligned} \tag{16.45}$$

The second and third integrals vanish, since we are integrating the cosine over its period. According to Eq. (16.4e), all terms in the fourth integral are zero when $m \neq n$. By evaluating the first integral and applying Eq. (16.4g) to the fourth integral for the case $m = n$, we obtain

$$\boxed{P = V_{\text{dc}} I_{\text{dc}} + \frac{1}{2} \sum_{n=1}^{\infty} V_n I_n \cos(\theta_n - \phi_n)} \tag{16.46}$$

This shows that in average-power calculation involving periodic voltage and current, the total average power is the sum of the average powers in each harmonically related voltage and current.

Given a periodic function $f(t)$, its rms value (or the effective value) is given by

$$F_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T f^2(t)\, dt} \tag{16.47}$$

Substituting $f(t)$ in Eq. (16.10) into Eq. (16.47) and noting that $(a + b)^2 = a^2 + 2ab + b^2$, we obtain

$$\begin{aligned} F_{\text{rms}}^2 &= \frac{1}{T} \int_0^T \left[a_0^2 + 2 \sum_{n=1}^{\infty} a_0 A_n \cos(n\omega_0 t + \phi_n) \right. \\ &\qquad \left. + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_n A_m \cos(n\omega_0 t + \phi_n) \cos(m\omega_0 t + \phi_m) \right] dt \\ &= \frac{1}{T} \int_0^T a_0^2\, dt + 2 \sum_{n=1}^{\infty} a_0 A_n \frac{1}{T} \int_0^T \cos(n\omega_0 t + \phi_n)\, dt \\ &\quad + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_n A_m \frac{1}{T} \int_0^T \cos(n\omega_0 t + \phi_n) \cos(m\omega_0 t + \phi_m)\, dt \end{aligned} \tag{16.48}$$

Distinct integers n and m have been introduced to handle the product of the two series summations. Using the same reasoning as above, we get

$$F_{\text{rms}}^2 = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2$$

or

$$F_{\text{rms}} = \sqrt{a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2} \tag{16.49}$$

In terms of Fourier coefficients a_n and b_n, Eq. (16.49) may be written as

$$F_{\text{rms}} = \sqrt{a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)} \tag{16.50}$$

If $f(t)$ is the current through a resistor R, then the power dissipated in the resistor is

$$P = R F_{\text{rms}}^2 \tag{16.51}$$

Or if $f(t)$ is the voltage across a resistor R, the power dissipated in the resistor is

$$P = \frac{F_{\text{rms}}^2}{R} \tag{16.52}$$

One can avoid specifying the nature of the signal by choosing a 1-Ω resistance. The power dissipated by the 1-Ω resistance is

$$\boxed{P_{1\Omega} = F_{\text{rms}}^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)} \tag{16.53}$$

Historical note: Named after the French mathematician Marc-Antoine Parseval Deschemes (1755–1836).

This result is known as *Parseval's theorem*. Notice that a_0^2 is the power in the dc component, while $1/2(a_n^2 + b_n^2)$ is the ac power in the nth harmonic. Thus, Parseval's theorem states that the average power in a periodic signal is the sum of the average power in its dc component and the average powers in its harmonics.

EXAMPLE 16.8

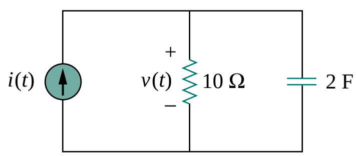


Figure 16.26 For Example 16.8.

Determine the average power supplied to the circuit in Fig. 16.26 if $i(t) = 2 + 10\cos(t + 10°) + 6\cos(3t + 35°)$ A.

Solution:

The input impedance of the network is

$$\mathbf{Z} = 10 \,\Big\|\, \frac{1}{j2\omega} = \frac{10(1/j2\omega)}{10 + 1/j2\omega} = \frac{10}{1 + j20\omega}$$

Hence,

$$\mathbf{V} = \mathbf{IZ} = \frac{10\mathbf{I}}{\sqrt{1 + 400\omega^2}\underline{/\tan^{-1} 20\omega}}$$

For the dc component, $\omega = 0$,

$$\mathbf{I} = 2\text{ A} \qquad \Longrightarrow \qquad \mathbf{V} = 10(2) = 20\text{ V}$$

This is expected, because the capacitor is an open circuit to dc and the entire 2-A current flows through the resistor. For $\omega = 1$ rad/s,

$$\mathbf{I} = 10\underline{/10°} \qquad \Longrightarrow \qquad \mathbf{V} = \frac{10(10\underline{/10°})}{\sqrt{1 + 400}\underline{/\tan^{-1} 20}} = 5\underline{/-77.14°}$$

For $\omega = 3$ rad/s,

$$\mathbf{I} = 6\underline{/45°} \qquad \Longrightarrow \qquad \mathbf{V} = \frac{10(6\underline{/45°})}{\sqrt{1 + 3600}\underline{/\tan^{-1} 60}} = 1\underline{/-44.05°}$$

Thus, in the time domain,

$$v(t) = 20 + 5\cos(t - 77.14°) + 1\cos(3t - 44.05°)\text{ V}$$

We obtain the average power supplied to the circuit by applying Eq. (16.46), as

$$P = V_{dc} I_{dc} + \frac{1}{2} \sum_{n=1}^{\infty} V_n I_n \cos(\theta_n - \phi_n)$$

To get the proper signs of θ_n and ϕ_n, we have to compare v and i in this example with Eqs. (16.42) and (16.43). Thus,

$$\begin{aligned} P &= 20(2) + \frac{1}{2}(5)(10)\cos[77.14^\circ - (-10^\circ)] \\ &\quad + \frac{1}{2}(1)(6)\cos[44.05^\circ - (-35^\circ)] \\ &= 40 + 1.247 + 0.05 = 41.5 \text{ W} \end{aligned}$$

Alternatively, we can find the average power absorbed by the resistor as

$$\begin{aligned} P &= \frac{V_{dc}^2}{R} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{|V_n|}{R} = \frac{20^2}{10} + \frac{1}{2} \cdot \frac{5^2}{10} + \frac{1}{2} \cdot \frac{1^2}{10} \\ &= 40 + 1.25 + 0.05 = 41.5 \text{ W} \end{aligned}$$

which is the same as the power supplied, since the capacitor absorbs no average power.

PRACTICE PROBLEM 16.8

The voltage and current at the terminals of a circuit are

$$\begin{aligned} v(t) &= 80 + 120\cos 120\pi t + 60\cos(360\pi t - 30^\circ) \\ i(t) &= 5\cos(120\pi t - 10^\circ) + 2\cos(360\pi t - 60^\circ) \end{aligned}$$

Find the average power absorbed by the circuit.

Answer: 347.4 W.

EXAMPLE 16.9

Find an estimate for the rms value of the voltage in Example 16.7.

Solution:

From Example 16.7, $v(t)$ is expressed as

$$\begin{aligned} v(t) = 1 &- 1.414\cos(t + 45^\circ) + 0.8944\cos(2t + 63.45^\circ) \\ &- 0.6345\cos(3t + 71.56^\circ) \\ &- 0.4851\cos(4t + 78.7^\circ) + \cdots \text{ V} \end{aligned}$$

Using Eq. (16.49),

$$\begin{aligned} V_{rms} &= \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2} \\ &= \sqrt{1^2 + \frac{1}{2}\left[(-1.414)^2 + (0.8944)^2 + (-0.6345)^2 + (-0.4851)^2 + \cdots\right]} \\ &= \sqrt{2.7186} = 1.649 \text{ V} \end{aligned}$$

This is only an estimate, as we have not taken enough terms of the series. The actual function represented by the Fourier series is

$$v(t) = \frac{\pi e^t}{\sinh \pi}, \qquad -\pi < t < \pi$$

with $v(t) = v(t + T)$. The exact rms value of this is 1.776 V.

PRACTICE PROBLEM 16.9

Find the rms value of the periodic current

$$i(t) = 8 + 30\cos 2t - 20\sin 2t + 15\cos 4t - 10\sin 4t \quad \text{A}$$

Answer: 29.61 A.

16.6 EXPONENTIAL FOURIER SERIES

A compact way of expressing the Fourier series in Eq. (16.3) is to put it in exponential form. This requires that we represent the sine and cosine functions in the exponential form using Euler's identity:

$$\cos n\omega_0 t = \frac{1}{2}[e^{jn\omega_0 t} + e^{-jn\omega_0 t}] \tag{16.54a}$$

$$\sin n\omega_0 t = \frac{1}{2j}[e^{jn\omega_0 t} - e^{-jn\omega_0 t}] \tag{16.54b}$$

Substituting Eq. (16.54) into Eq. (16.3) and collecting terms, we obtain

$$f(t) = a_0 + \frac{1}{2}\sum_{n=1}^{\infty}[(a_n - jb_n)e^{jn\omega_0 t} + (a_n + jb_n)e^{-jn\omega_0 t}] \tag{16.55}$$

If we define a new coefficient c_n so that

$$c_0 = a_0, \qquad c_n = \frac{(a_n - jb_n)}{2}, \qquad c_{-n} = c_n^* = \frac{(a_n + jb_n)}{2} \tag{16.56}$$

then $f(t)$ becomes

$$f(t) = c_0 + \sum_{n=1}^{\infty}(c_n e^{jn\omega_0 t} + c_{-n}e^{-jn\omega_0 t}) \tag{16.57}$$

or

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}} \tag{16.58}$$

This is the *complex* or *exponential Fourier series* representation of $f(t)$. Note that this exponential form is more compact than the sine-cosine form in Eq. (16.3). Although the exponential Fourier series coefficients c_n can also be obtained from a_n and b_n using Eq. (16.56), they can also be obtained directly from $f(t)$ as

$$\boxed{c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}\,dt} \tag{16.59}$$

where $\omega_0 = 2\pi/T$, as usual. The plots of the magnitude and phase of c_n versus $n\omega_0$ are called the *complex amplitude spectrum* and *complex phase spectrum* of $f(t)$, respectively. The two spectra form the complex frequency spectrum of $f(t)$.

The exponential Fourier series of a periodic function $f(t)$ describes the spectrum of $f(t)$ in terms of the amplitude and phase angle of ac components at positive and negative harmonic frequencies.

The coefficients of the three forms of Fourier series (sine-cosine form, amplitude-phase form, and exponential form) are related by

$$\boxed{A_n \underline{/\phi_n} = a_n - jb_n = 2c_n} \tag{16.60}$$

or

$$c_n = |c_n| \underline{/\theta_n} = \frac{\sqrt{a_n^2 + b_n^2}}{2} \underline{/-\tan^{-1} b_n/a_n} \tag{16.61}$$

if only $a_n > 0$. Note that the phase θ_n of c_n is equal to ϕ_n.

In terms of the Fourier complex coefficients c_n, the rms value of a periodic signal $f(t)$ can be found as

$$\begin{aligned} F_{\mathrm{rms}}^2 &= \frac{1}{T}\int_0^T f^2(t)\,dt = \frac{1}{T}\int_0^T f(t)\left[\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}\right] dt \\ &= \sum_{n=-\infty}^{\infty} c_n \left[\frac{1}{T}\int_0^T f(t) e^{jn\omega_0 t}\right] \\ &= \sum_{n=-\infty}^{\infty} c_n c_n^* = \sum_{n=-\infty}^{\infty} |c_n|^2 \end{aligned} \tag{16.62}$$

or

$$F_{\mathrm{rms}} = \sqrt{\sum_{n=-\infty}^{\infty} |c_n|^2} \tag{16.63}$$

Equation (16.62) can be written as

$$F_{\mathrm{rms}}^2 = |c_0|^2 + 2\sum_{n=1}^{\infty} |c_n|^2 \tag{16.64}$$

Again, the power dissipated by a 1-Ω resistance is

$$P_{1\Omega} = F_{\mathrm{rms}}^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{16.65}$$

which is a restatement of Parseval's theorem. The *power spectrum* of the signal $f(t)$ is the plot of $|c_n|^2$ versus $n\omega_0$. If $f(t)$ is the voltage across a resistor R, the average power absorbed by the resistor is F_{rms}^2/R; if $f(t)$ is the current through R, the power is $F_{\mathrm{rms}}^2 R$.

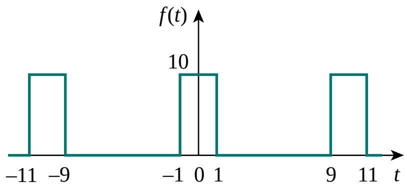


Figure 16.27 The periodic pulse train.

As an illustration, consider the periodic pulse train of Fig. 16.27. Our goal is to obtain its amplitude and phase spectra. The period of the pulse train is $T = 10$, so that $\omega_0 = 2\pi/T = \pi/5$. Using Eq. (16.59),

$$\begin{aligned} c_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt = \frac{1}{10}\int_{-1}^{1} 10e^{-jn\omega_0 t}\,dt \\ &= \frac{1}{-jn\omega_0}e^{-jn\omega_0 t}\Big|_{-1}^{1} = \frac{1}{-jn\omega_0}(e^{-jn\omega_0} - e^{jn\omega_0}) \\ &= \frac{2}{n\omega_0}\frac{e^{jn\omega_0} - e^{-jn\omega_0}}{2j} = 2\frac{\sin n\omega_0}{n\omega_0}, \qquad \omega_0 = \frac{\pi}{5} \\ &= 2\frac{\sin n\pi/5}{n\pi/5} \end{aligned} \tag{16.66}$$

and

$$f(t) = 2\sum_{n=-\infty}^{\infty} \frac{\sin n\pi/5}{n\pi/5}e^{jn\pi t/5} \tag{16.67}$$

Notice from Eq. (16.66) that c_n is the product of 2 and a function of the form $\sin x/x$. This function is known as the *sinc function*; we write it as

The sinc function is called the *sampling function* in communication theory, where it is very useful.

$$\operatorname{sinc}(x) = \frac{\sin x}{x} \tag{16.68}$$

Some properties of the sinc function are important here. For zero argument, the value of the sinc function is unity,

$$\operatorname{sinc}(0) = 1 \tag{16.69}$$

This is obtained applying L'Hopital's rule to Eq. (16.68). For an integral multiple of π, the value of the sinc function is zero,

$$\operatorname{sinc}(n\pi) = 0, \qquad n = 1, 2, 3, \ldots \tag{16.70}$$

Also, the sinc function shows even symmetry. With all this in mind, we can obtain the amplitude and phase spectra of $f(t)$. From Eq. (16.66), the magnitude is

$$|c_n| = 2\left|\frac{\sin n\pi/5}{n\pi/5}\right| \tag{16.71}$$

while the phase is

$$\theta_n = \begin{cases} 0^\circ, & \sin\dfrac{n\pi}{5} > 0 \\ 180^\circ, & \sin\dfrac{n\pi}{5} < 0 \end{cases} \tag{16.72}$$

Figure 16.28 shows the plot of $|c_n|$ versus n for n varying from -10 to 10, where $n = \omega/\omega_0$ is the normalized frequency. Figure 16.29 shows the plot of θ_n versus n. Both the amplitude spectrum and phase spectrum are called *line spectra*, because the value of $|c_n|$ and θ_n occur only at discrete values of frequencies. The spacing between the lines is ω_0. The power spectrum, which is the plot of $|c_n|^2$ versus $n\omega_0$, can also be plotted. Notice that the sinc function forms the envelope of the amplitude spectrum.

Examining the input and output spectra allows visualization of the effect of a circuit on a periodic signal.

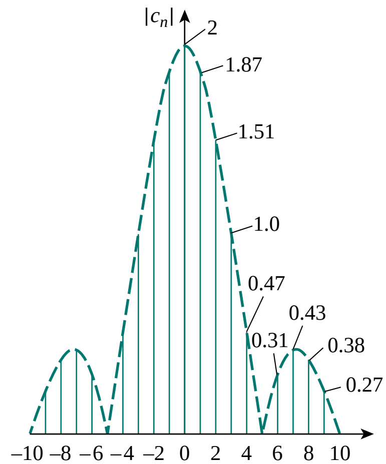


Figure 16.28 The amplitude of a periodic pulse train.

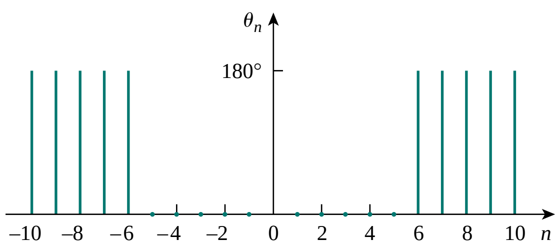


Figure 16.29 The phase spectrum of a periodic pulse train.

EXAMPLE 16.10

Find the exponential Fourier series expansion of the periodic function $f(t) = e^t, 0 < t < 2\pi$ with $f(t + 2\pi) = f(t)$.

Solution:

Since $T = 2\pi$, $\omega_0 = 2\pi/T = 1$. Hence,

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^t e^{-jnt}\,dt$$

$$= \frac{1}{2\pi}\frac{1}{1-jn}e^{(1-jn)t}\Big|_0^{2\pi} = \frac{1}{2\pi(1-jn)}[e^{2\pi}e^{-j2\pi n} - 1]$$

But by Euler's identity,

$$e^{-j2\pi n} = \cos 2\pi n - j\sin 2\pi n = 1 - j0 = 1$$

Thus,

$$c_n = \frac{1}{2\pi(1-jn)}[e^{2\pi} - 1] = \frac{85}{1-jn}$$

The complex Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{85}{1-jn}e^{jnt}$$

We may want to plot the complex frequency spectrum of $f(t)$. If we let $c_n = |c_n|\underline{/\theta_n}$, then

$$|c_n| = \frac{85}{\sqrt{1+n^2}}, \qquad \theta_n = \tan^{-1} n$$

By inserting in negative and positive values of n, we obtain the amplitude and the phase plots of c_n versus $n\omega_0 = n$, as in Fig. 16.30.

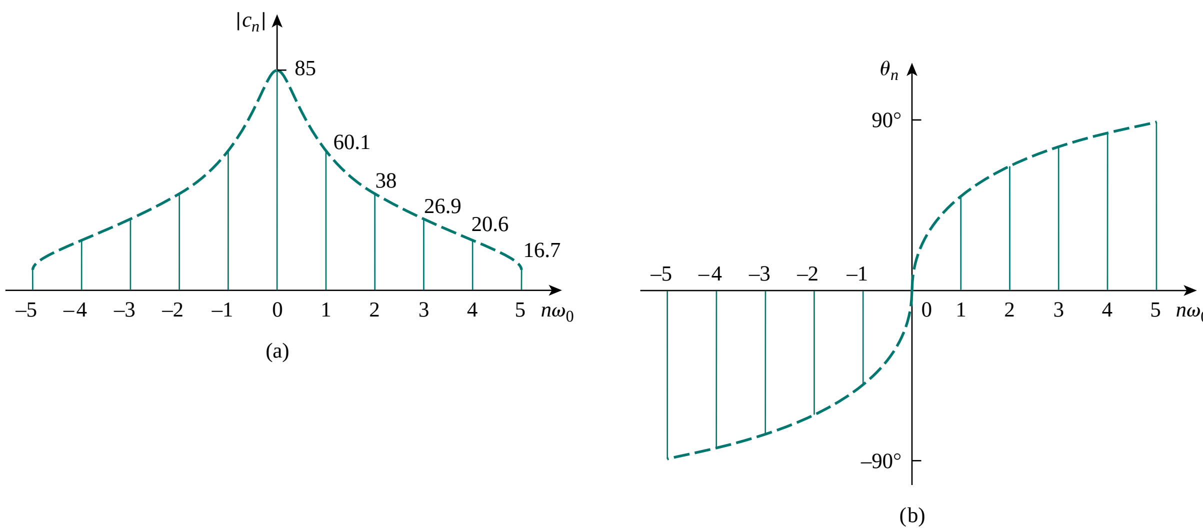


Figure 16.30 The complex frequency spectrum of the function in Example 16.10: (a) amplitude spectrum, (b) phase spectrum.

PRACTICE PROBLEM 16.10

Obtain the complex Fourier series of the function in Fig. 16.1.

Answer: $f(t) = \frac{1}{2} - \sum_{\substack{n=-\infty \\ n\neq 0 \\ n=\text{odd}}}^{\infty} \frac{j}{n\pi} e^{jn\pi t}$.

EXAMPLE 16.11

Find the complex Fourier series of the sawtooth wave in Fig. 16.9. Plot the amplitude and the phase spectra.

Solution:

From Fig. 16.9, $f(t) = t, 0 < t < 1, T = 1$ so that $\omega_0 = 2\pi/T = 2\pi$. Hence,

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}\,dt = \frac{1}{1}\int_0^1 te^{-j2n\pi t}\,dt \tag{16.11.1}$$

But

$$\int te^{at}\,dt = \frac{e^{at}}{a^2}(ax-1)+C$$

Applying this to Eq. (16.11.1) gives

$$\begin{aligned} c_n &= \frac{e^{-j2n\pi t}}{(-j2n\pi)^2}(-j2n\pi t-1)\Big|_0^1 \\ &= \frac{e^{-j2n\pi}(-j2n\pi-1)+1}{-4n^2\pi^2} \end{aligned} \tag{16.11.2}$$

Again,

$$e^{-j2\pi n} = \cos 2\pi n - j\sin 2\pi n = 1 - j0 = 1$$

so that Eq. (16.11.2) becomes

$$c_n = \frac{-j2n\pi}{-4n^2\pi^2} = \frac{j}{2n\pi} \tag{16.11.3}$$

This does not include the case when $n = 0$. When $n = 0$,

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{1}\int_0^1 t\,dt = \frac{t^2}{2}\Big|_1^0 = 0.5 \tag{16.11.4}$$

Hence,

$$f(t) = 0.5 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{j}{2n\pi}e^{j2n\pi t} \tag{16.11.5}$$

and

$$|c_n| = \begin{cases} \dfrac{1}{2|n|\pi}, & n \neq 0 \\ 0.5, & n = 0 \end{cases}, \qquad \theta_n = 90^\circ, \quad n \neq 0 \tag{16.11.6}$$

By plotting $|c_n|$ and θ_n for different n, we obtain the amplitude spectrum and the phase spectrum shown in Fig. 16.31.

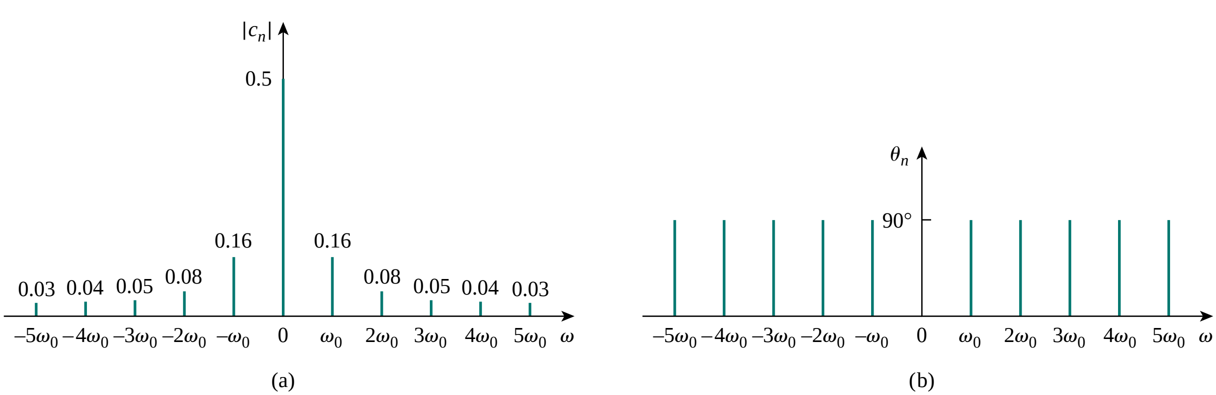


Figure 16.31 For Example 16.11: (a) amplitude spectrum, (b) phase spectrum.

PRACTICE PROBLEM 16.11

Obtain the complex Fourier series expansion of $f(t)$ in Fig. 16.17. Show the amplitude and phase spectra.

Answer: $f(t) = -\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{j(-1)^n}{n\pi} e^{jn\pi t}$. See Fig. 16.32 for the spectra.

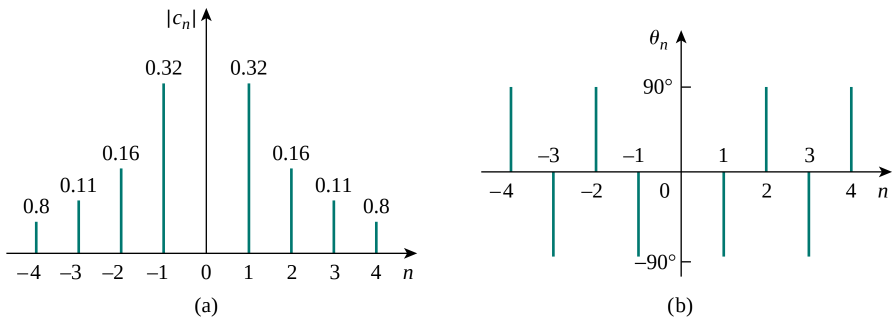


Figure 16.32 For Practice Prob. 16.11: (a) amplitude spectrum, (b) phase spectrum.

16.7 FOURIER ANALYSIS WITH *PSPICE*

Fourier analysis is usually performed with *PSpice* in conjunction with transient analysis. Therefore, we must do a transient analysis in order to perform a Fourier analysis.

To perform the Fourier analysis of a waveform, we need a circuit whose input is the waveform and whose output is the Fourier decomposition. A suitable circuit is a current (or voltage) source in series with a 1-Ω resistor as shown in Fig. 16.33. The waveform is inputted as $v_s(t)$ using VPULSE for a pulse or VSIN for a sinusoid, and the attributes of the waveform are set over its period T. The output V(1) from node 1 is the dc level (a_0) and the first nine harmonics (A_n) with their corresponding phases ψ_n; that is,

$$v_o(t) = a_0 + \sum_{n=1}^{9} A_n \sin(n\omega_0 t + \psi_n) \tag{16.73}$$

where

$$A_n = \sqrt{a_n^2 + b_n^2}, \qquad \psi_n = \phi_n - \frac{\pi}{2}, \qquad \phi_n = \tan^{-1}\frac{b_n}{a_n} \tag{16.74}$$

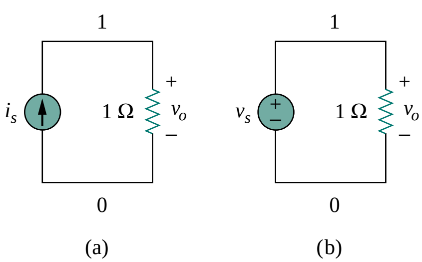


Figure 16.33 Fourier analysis with *PSpice* using: (a) a current source, (b) a voltage source.

Notice in Eq. (16.74) that the *PSpice* output is in the sine and angle form rather than the cosine and angle form in Eq. (16.10). The *PSpice* output also includes the normalized Fourier coefficients. Each coefficient a_n is normalized by dividing it by the magnitude of the fundamental a_1 so that the normalized component is a_n/a_1. The corresponding phase ψ_n is normalized by subtracting from it the phase ψ_1 of the fundamental, so that the normalized phase is $\psi_n - \psi_1$.

There are two types of Fourier analyses offered by *PSpice for Windows: Discrete Fourier Transform* (DFT) performed by the *PSpice* program and *Fast Fourier Transform* (FFT) performed by the *Probe* program. While DFT is an approximation of the exponential Fourier series, FTT is an algorithm for rapid efficient numerical computation of DFT. A full discussion of DFT and FTT is beyond the scope of this book.

16.7.1 Discrete Fourier Transform

A discrete Fourier transform (DFT) is performed by the *PSpice* program, which tabulates the harmonics in an output file. To enable a Fourier analysis, we select **Analysis/Setup/Transient** and bring up the Transient dialog box, shown in Fig. 16.34. The *Print Step* should be a small fraction of the period T, while the *Final Time* could be $6T$. The *Center Frequency* is the fundamental frequency $f_0 = 1/T$. The particular variable whose DFT is desired, V(1) in Fig. 16.34, is entered in the **Output Vars** command box. In addition to filling in the Transient dialog box, **DCLICK** *Enable Fourier*. With the Fourier analysis enabled and the schematic saved, run *PSpice* by selecting **Analysis/Simulate** as usual. The program executes a harmonic decomposition into Fourier components of the result of the transient analysis. The results are sent to an output file which you can retrieve by selecting **Analysis/Examine Output**. The output file includes the dc value and the first nine harmonics by default, although you can specify more in the *Number of harmonics* box (see Fig. 16.34).

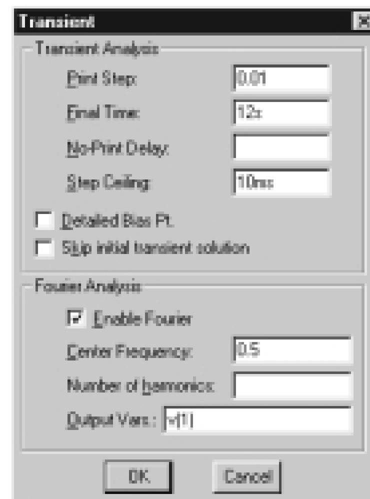


Figure 16.34 Transient dialog box.

16.7.2 Fast Fourier Transform

A fast Fourier transform (FFT) is performed by the *Probe* program and displays as a *Probe* plot the complete spectrum of a transient expression. As explained above, we first construct the schematic in Fig. 16.33(b) and enter the attributes of the waveform. We also need to enter the *Print Step* and the *Final Time* in the Transient dialog box. Once this is done, we can obtain the FFT of the waveform in two ways.

One way is to insert a voltage marker at node 1 in the schematic of the circuit in Fig. 16.33(b). After saving the schematic and selecting **Analysis/Simulate**, the waveform V(1) will be displayed in the Probe window. Double clicking the FFT icon in the Probe menu will automatically replace the waveform with its FFT. From the FFT-generated graph, we can obtain the harmonics. In case the FFT-generated graph is crowded, we can use the *User Defined* data range (see Fig. 16.35) to specify a smaller range.

Another way of obtaining the FFT of V(1) is to not insert a voltage marker at node 1 in the schematic. After selecting **Analysis/Simulate**, the Probe window will come up with no graph on it. We select **Trace/Add** and type V(1) in the **Trace Command** box and **DCLICKL OK**. We now select **Plot/X-Axis Settings** to bring up the *X Axis Setting* dialog box shown in Fig. 16.35 and then select **Fourier/OK**. This will cause the FFT of the selected trace (or traces) to be displayed. This second approach is useful for obtaining the FFT of any trace associated with the circuit.

A major advantage of the FFT method is that it provides graphical output. But its major disadvantage is that some of the harmonics may be too small to see.

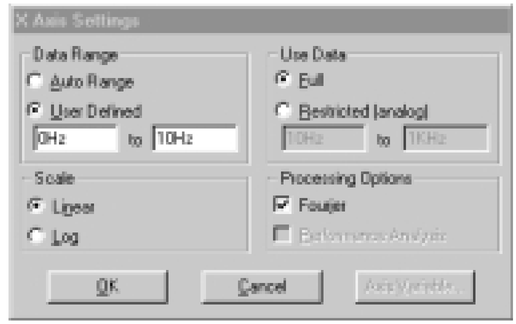


Figure 16.35 *X* axis settings dialog box.

In both DFT and FFT, we should let the simulation run for a large number of cycles and use a small value of *Step Ceiling* (in the Transient dialog box) to ensure accurate results. The *Final Time* in the Transient dialog box should be at least five times the period of the signal to allow the simulation to reach steady state.

EXAMPLE 16.12

Use *PSpice* to determine the Fourier coefficients of the signal in Fig. 16.1.

Solution:

Figure 16.36 shows the schematic for obtaining the Fourier coefficients. With the signal in Fig. 16.1 in mind, we enter the attributes of the voltage source VPULSE as shown in Fig. 16.36. We will solve this example using both the DFT and FFT approaches.

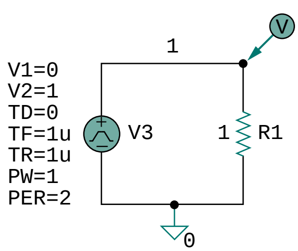


Figure 16.36 Schematic for Example 16.12.

METHOD 1 **DFT Approach:** (The voltage marker in Fig. 16.36 is not needed for this method.) From Fig. 16.1, it is evident that $T = 2$ s,

$$f_0 = \frac{1}{T} = \frac{1}{2} = 0.5 \text{ Hz}$$

So, in the transient dialog box, we select the *Final Time* as 6T = 12 s, the *Print Step* as 0.01 s, the *Step Ceiling* as 10 ms, the *Center Frequency* as 0.5 Hz, and the output variable as V(1). (In fact, Fig. 16.34 is for this particular example.) When *PSpice* is run, the output file contains the following result.

```
FOURIER COEFFICIENTS OF TRANSIENT RESPONSE V(1)

DC COMPONENT = 4.989950E-01

HARMONIC  FREQUENCY  FOURIER    NORMALIZED    PHASE     NORMALIZED
  NO        (HZ)     COMPONENT  COMPONENT     (DEG)     PHASE (DEG)

  1       5.000E-01  6.366E-01  1.000E+00  -1.809E-01   0.000E+00
  2       1.000E+00  2.012E-03  3.160E-03  -9.226E+01  -9.208E+01
  3       1.500E+00  2.122E-01  3.333E-01  -5.427E-01  -3.619E-01
```

(continued)

(continued)

4	2.000E+00	2.016E-03	3.167E-03	-9.451E+01	-9.433E+01
5	2.500E+00	1.273E-01	1.999E-01	-9.048E-01	-7.239E-01
6	3.000E+00	2.024E-03	3.180E-03	-9.676E+01	-9.658E+01
7	3.500E+00	9.088E-02	1.427E-01	-1.267E+00	-1.086E+00
8	4.000E+00	2.035E-03	3.197E-03	-9.898E+01	-9.880E+01
9	4.500E+00	7.065E-02	1.110E-01	-1.630E+00	-1.449E+00

Comparing the result with that in Eq. (16.1.7) (see Example 16.1) or with the spectra in Fig. 16.4 shows a close agreement. From Eq. (16.1.7), the dc component is 0.5 while *PSpice* gives 0.498995. Also, the signal has only odd harmonics with phase $\psi_n = -90^\circ$, whereas *PSpice* seems to indicate that the signal has even harmonics although the magnitudes of the even harmonics are small.

METHOD 2 **FFT Approach:** With voltage marker in Fig. 16.36 in place, we run *PSpice* and obtain the waveform V(1) shown in Fig. 16.37(a) on the Probe window. By double clicking the FFT icon in the Probe menu and changing the X-axis setting to 0 to 10 Hz, we obtain the FFT of V(1) as shown in Fig. 16.37(b). The FFT-generated graph contains the dc and harmonic components within the selected frequency range. Notice that the magnitudes and frequencies of the harmonics agree with the DFT-generated tabulated values.

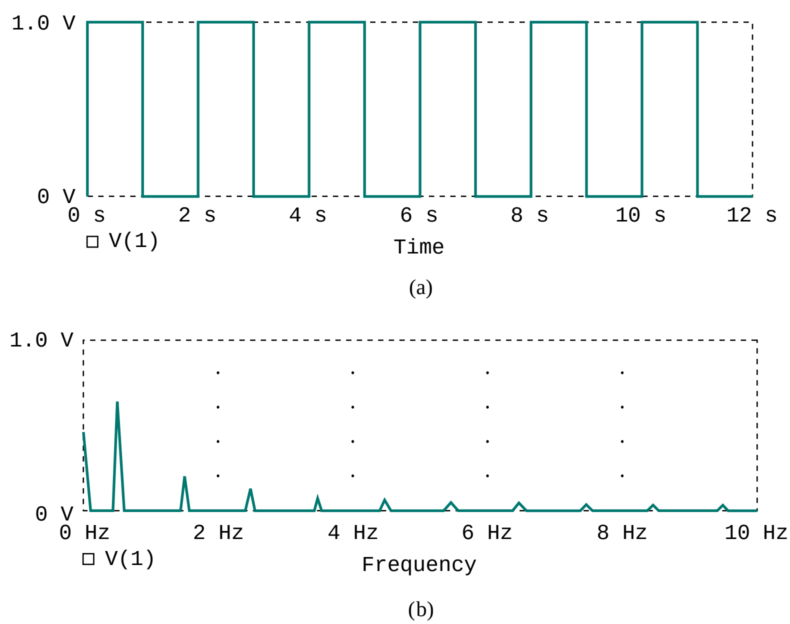


Figure 16.37 (a) Original waveform of Fig. 16.1, (b) FFT of the waveform.

PRACTICE PROBLEM 16.12

Obtain the Fourier coefficients of the function in Fig. 16.7 using *PSpice*.

Answer:

```
FOURIER COEFFICIENTS OF TRANSIENT RESPONSE V(1)

DC COMPONENT = 4.950000E-01
```

HARMONIC NO	FREQUENCY (HZ)	FOURIER COMPONENT	NORMALIZED COMPONENT	PHASE (DEG)	NORMALIZED PHASE (DEG)
1	1.000E+00	3.184E-01	1.000E+00	-1.782E+02	0.000E+00
2	2.000E+00	1.593E-01	5.002E-01	-1.764E+02	1.800E+00
3	3.000E+00	1.063E-01	3.338E-01	-1.746E+02	3.600E+00
4	4.000E+00	7.979E-02	2.506E-03	-1.728E+02	5.400E+00
5	5.000E+00	6.392E-01	2.008E-01	-1.710E+02	7.200E+00
6	6.000E+00	5.337E-02	1.676E-03	-1.692E+02	9.000E+00
7	7.000E+00	4.584E-02	1.440E-01	-1.674E+02	1.080E+01
8	8.000E+00	4.021E-02	1.263E-01	-1.656E+02	1.260E+01
9	9.000E+00	3.584E-02	1.126E-01	-1.638E+02	1.440E+01

EXAMPLE 16.13

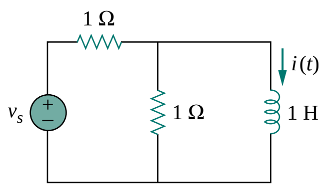


Figure 16.38 For Example 16.13.

If v_s in the circuit of Fig. 16.38 is a sinusoidal voltage source of amplitude 12 V and frequency 100 Hz, find current $i(t)$.

Solution:

The schematic is shown in Fig. 16.39. We may use the DFT approach to obtain the Fourier coefficents of $i(t)$. Since the period of the input waveform is $T = 1/100 = 10$ ms, in the Transient dialog box we select *Print Step:* 0.1 ms, *Final Time:* 100 ms, *Center Frequency:* 100 Hz, *Number of harmonics:* 4, and *Output Vars:* I(L1). When the circuit is simulated, the output file includes the following.

```
FOURIER COEFFICIENTS OF TRANSIENT RESPONSE I(VD)

DC COMPONENT = 8.583269E-03
```

HARMONIC NO	FREQUENCY (HZ)	FOURIER COMPONENT	NORMALIZED COMPONENT	PHASE (DEG)	NORMALIZED PHASE (DEG)
1	1.000E+02	8.730E-03	1.000E+00	-8.984E+01	0.000E+00
2	2.000E+02	1.017E-04	1.165E-02	-8.306E+01	6.783E+00
3	3.000E+02	6.811E-05	7.802E-03	-8.235E+01	7.490E+00
4	4.000E+02	4.403E-05	5.044E-03	-8.943E+01	4.054E+00

With the Fourier coefficients, the Fourier series describing the current $i(t)$ can be obtained using Eq. (16.73); that is,

$$\begin{aligned} i(t) &= 8.5833 + 8.73\sin(2\pi\cdot 100t - 89.84^\circ) \\ &\quad + 0.1017\sin(2\pi\cdot 200t - 83.06^\circ) \\ &\quad + 0.068\sin(2\pi\cdot 300t - 82.35^\circ) + \cdots \text{ mA} \end{aligned}$$

We can also use the FFT approach to cross-check our result. The current marker is inserted at pin 1 of the inductor as shown in Fig. 16.39. Running *PSpice* will automatically produce the plot of I(L1) in the Probe window, as shown in Fig. 16.40(a). By double clicking the FFT icon and setting the range of the X-axis from 0 to 200 Hz, we generate the FFT of I(L1) shown in Fig. 16.40(b). It is clear from the FFT-generated plot that only the dc component and the first harmonic are visible. Higher harmonics are negligibly small.

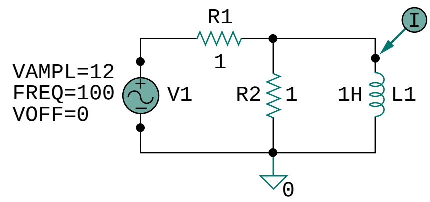


Figure 16.39 Schematic of the circuit in Fig. 16.38.

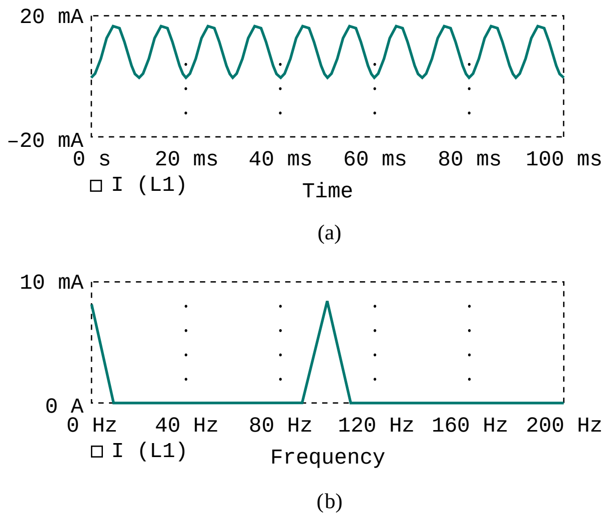


Figure 16.40 For Example 16.13: (a) plot of $i(t)$, (b) the FFT of $i(t)$.

PRACTICE PROBLEM 16.13

A sinusoidal current source of amplitude 4 A and frequency 2 kHz is applied to the circuit in Fig. 16.41. Use *PSpice* to find $v(t)$.

Answer: $v(t) = -150.72 + 145.5 \sin(4\pi \cdot 10^3 t + 90°) + \cdots \ \mu$V. The Fourier components are shown below.

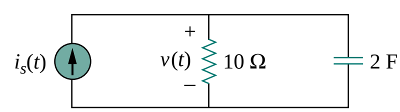


Figure 16.41 For Practice Prob. 16.14.

```
FOURIER COEFFICIENTS OF TRANSIENT RESPONSE V(R1:1)

DC COMPONENT =  -1.507169E-04

HARMONIC  FREQUENCY   FOURIER    NORMALIZED     PHASE     NORMALIZED
   NO       (HZ)     COMPONENT   COMPONENT      (DEG)     PHASE (DEG)

   1      2.000E+03  1.455E-04   1.000E+00    9.006E+01    0.000E+00
   2      4.000E+03  1.851E-06   1.273E-02    9.597E+01    5.910E+00
   3      6.000E+03  1.406E-06   9.662E-03    9.323E+01    3.167E+00
   4      8.000E+03  1.010E-06   6.946E-02    8.077E+01   -9.292E+00
```

†16.8 APPLICATIONS

We demonstrated in Section 16.4 that the Fourier series expansion permits the application of the phasor techniques used in ac analysis to circuits containing nonsinusoidal periodic excitations. The Fourier series has many other practical applications, particularly in communications and signal processing. Typical applications include spectrum analysis, filtering, rectification, and harmonic distortion. We will consider two of these: spectrum analyzers and filters.

16.8.1 Spectrum Analyzers

The Fourier series provides the spectrum of a signal. As we have seen, the spectrum consists of the amplitudes and phases of the harmonics versus frequency. By providing the spectrum of a signal $f(t)$, the Fourier series helps us identify the pertinent features of the signal. It demonstrates which frequencies are playing an important role in the shape of the output and which ones are not. For example, audible sounds have significant components in the frequency range of 20 Hz to 15 kHz, while visible light signals range from 10^5 GHz to 10^6 GHz. Table 16.4 presents some other signals and the frequency ranges of their components. A periodic function is said to be *band-limited* if its amplitude spectrum contains only a finite number of coefficients A_n or c_n. In this case, the Fourier series becomes

TABLE 16.4 Frequency ranges of typical signals.

Signal	Frequency Range
Audible sounds	20 Hz to 15 kHz
AM radio	540–1600 kHz
Short-wave radio	3–36 MHz
Video signals (U.S. standards)	dc to 4.2 MHz
VHF television, FM radio	54–216 MHz
UHF television	470–806 MHz
Cellular telephone	824–891.5 MHz
Microwaves	2.4–300 GHz
Visible light	10^5–10^6 GHz
X-rays	10^8–10^9 GHz

$$f(t) = \sum_{n=-N}^{N} c_n e^{jn\omega_0 t} = a_0 + \sum_{n=1}^{N} A_n \cos(n\omega_0 t + \phi_n) \qquad (16.75)$$

This shows that we need only $2N+1$ terms (namely, $a_0, A_1, A_2, \dots, A_N, \phi_1, \phi_2, \dots, \phi_N$) to completely specify $f(t)$ if ω_0 is known. This leads to the *sampling theorem:* a band-limited periodic function whose Fourier series contains N harmonics is uniquely specified by its values at $2N+1$ instants in one period.

A *spectrum analyzer* is an instrument that displays the amplitude of the components of a signal versus frequency. In other words, it shows the various frequency components (spectral lines) that indicate the amount of energy at each frequency. It is unlike an oscilloscope, which displays the entire signal (all components) versus time. An oscilloscope shows the signal in the time domain, while the spectrum analyzer shows the signal in the frequency domain. There is perhaps no instrument more useful to a circuit analyst than the spectrum analyzer. An analyzer can conduct noise and spurious signal analysis, phase checks, electromagnetic interference and filter examinations, vibration measurements, radar measurements, and more. Spectrum analyzers are commercially available in various sizes and shapes. Figure 16.42 displays a typical one.

16.8.2 Filters

Filters are an important component of electronics and communications systems. Chapter 14 presented a full discussion on passive and active filters. Here, we investigate how to design filters to select the fundamental component (or any desired harmonic) of the input signal and reject other harmonics. This filtering process cannot be accomplished without the

Figure 16.42 A typical spectrum analyzer. *(Courtesy of Hewlett-Packer.)*

Fourier series expansion of the input signal. For the purpose of illustration, we will consider two cases, a lowpass filter and a bandpass filter. In Example 16.6, we already looked at a highpass RL filter.

The output of a lowpass filter depends on the input signal, the transfer function $H(\omega)$ of the filter, and the corner or half-power frequency ω_c. We recall that $\omega_c = 1/RC$ for an RC passive filter. As shown in Fig. 16.43(a), the lowpass filter passes the dc and low-frequency components, while blocking the high-frequency components. By making ω_c sufficiently large ($\omega_c \gg \omega_0$, e.g., making C small), a large number of the

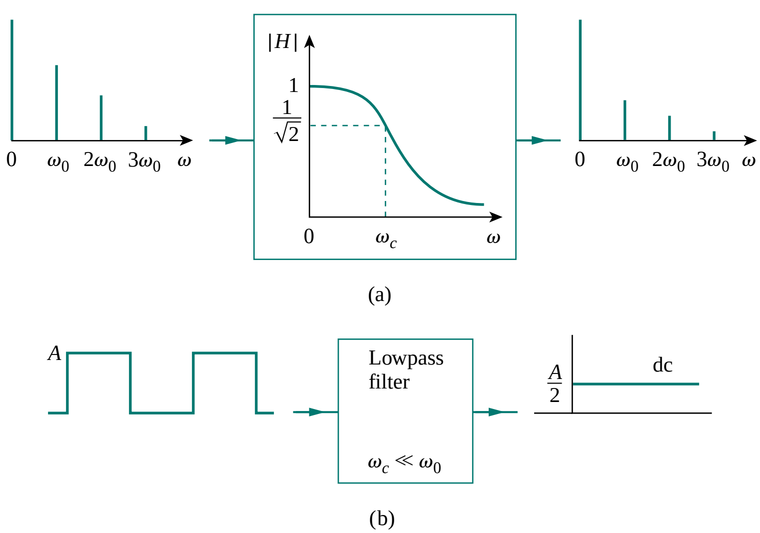


Figure 16.43 (a) Input and output spectra of a lowpass filter, (b) the lowpass filter passes only the dc component when $\omega_c \ll \omega_0$.

harmonics can be passed. On the other hand, by making ω_c sufficiently small ($\omega_c \ll \omega_0$), we can block out all the ac components and pass only dc, as shown typically in Fig. 16.43(b). (See Fig. 16.2(a) for the Fourier series expansion of the square wave.)

In this section, we have used ω_c for the center frequency of the bandpass filter instead of ω_0 as in Chapter 14, to avoid confusing ω_0 with the fundamental frequency of the input signal.

Similarly, the output of a bandpass filter depends on the input signal, the transfer function of the filter $H(\omega)$, its bandwidth B, and its center frequency ω_c. As illustrated in Fig. 16.44(a), the filter passes all the harmonics of the input signal within a band of frequencies ($\omega_1 < \omega < \omega_2$) centered around ω_c. We have assumed that ω_0, $2\omega_0$, and $3\omega_0$ are within that band. If the filter is made highly selective ($B \ll \omega_0$) and $\omega_c = \omega_0$, where ω_0 is the fundamental frequency of the input signal, the filter passes only the fundamental component ($n = 1$) of the input and blocks out all higher harmonics. As shown in Fig. 16.44(b), with a square wave as input, we obtain a sine wave of the same frequency as the output. (Again, refer to Fig. 16.2(a).)

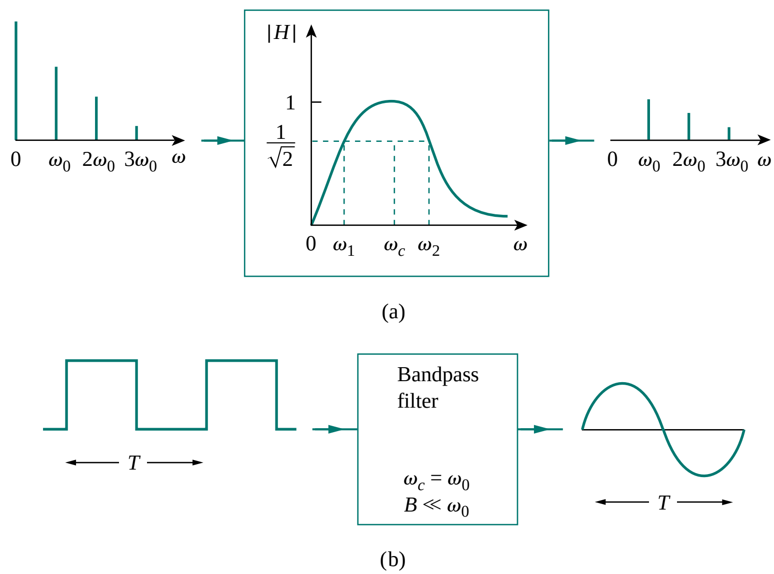


Figure 16.44 (a) Input and output spectra of a bandpass filter, (b) the bandpass filter passes only the fundamental component when $B \ll \omega_0$.

EXAMPLE 16.14

If the sawtooth waveform in Fig. 16.45(a) is applied to an ideal lowpass filter with the transfer function shown in Fig. 16.45(b), determine the output.

Solution:

The input signal in Fig. 16.45(a) is the same as the signal in Fig. 16.9. From Practice Prob. 16.2, we know that the Fourier series expansion is

$$x(t) = \frac{1}{2} - \frac{1}{\pi}\sin\omega_0 t - \frac{1}{2\pi}\sin 2\omega_0 t - \frac{1}{3\pi}\sin 3\omega_0 t - \cdots$$

where the period is $T = 1$ s and the fundamental frequency is $\omega_0 = 2\pi$ rad/s. Since the corner frequency of the filter is $\omega_c = 10$ rad/s, only the dc component and harmonics with $n\omega_0 < 10$ will be passed. For $n = 2$, $n\omega_0 = 4\pi = 12.566$ rad/s, which is higher than 10 rad/s, meaning that second and higher harmonics will be rejected. Thus, only the dc and fundamental components will be passed. Hence the output of the filter is

$$y(t) = \frac{1}{2} - \frac{1}{\pi}\sin 2\pi t$$

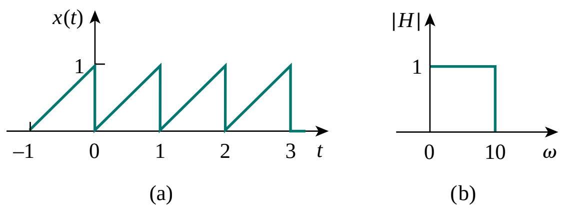


Figure 16.45 For Example 16.14.

PRACTICE PROBLEM 16.14

Rework Example 16.14 if the lowpass filter is replaced by the ideal bandpass filter shown in Fig. 16.46.

Answer: $y(t) = -\frac{1}{3\pi}\sin 3\omega_0 t - \frac{1}{4\pi}\sin 4\omega_0 t - \frac{1}{5\pi}\sin 5\omega_0 t.$

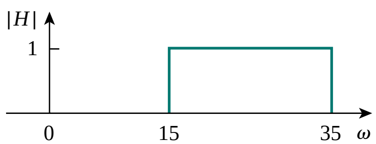


Figure 16.46 For Practice Prob. 16.14.

16.9 SUMMARY

1. A periodic function is one that repeats itself every T seconds; that is, $f(t \pm nT) = f(t), n = 1, 2, 3, \ldots$.
2. Any nonsinusoidal periodic function $f(t)$ that we encounter in electrical engineering can be expressed in terms of sinusoids using Fourier series:

$$f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}}$$

where $\omega_0 = 2\pi/T$ is the fundamental frequency. The Fourier series resolves the function into the dc component a_0 and an ac component containing infinitely many harmonically related sinusoids. The

Fourier coefficients are determined as

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt, \qquad a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt$$

If $f(t)$ is an even function, $b_n = 0$, and when $f(t)$ is odd, $a_0 = 0$ and $a_n = 0$. If $f(t)$ is half-wave symmetric, $a_0 = a_n = b_n = 0$ for even values of n.

3. An alternative to the trigonometric (or sine-cosine) Fourier series is the amplitude-phase form

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

where

$$A_n = \sqrt{a_n^2 + b_n^2}, \qquad \phi_n = -\tan^{-1}\frac{b_n}{a_n}$$

4. Fourier series representation allows us to apply the phasor method in analyzing circuits when the source function is a nonsinusoidal periodic function. We use phasor technique to determine the response of each harmonic in the series, transform the responses to the time domain, and add them up.
5. The average-power of periodic voltage and current is

$$P = V_{\text{dc}} I_{\text{dc}} + \frac{1}{2}\sum_{n=1}^{\infty} V_n I_n \cos(\theta_n - \phi_n)$$

In other words, the total average power is the sum of the average powers in each harmonically related voltage and current.

6. A periodic function can also be represented in terms of an exponential (or complex) Fourier series as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where

$$c_n = \frac{1}{T}\int_0^T f(t) e^{-jn\omega_0 t}\,dt$$

and $\omega_0 = 2\pi/T$. The exponential form describes the spectrum of $f(t)$ in terms of the amplitude and phase of ac components at positive and negative harmonic frequencies. Thus, there are three basic forms of Fourier series representation: the trigonometric form, the amplitude-phase form, and the exponential form.

7. The frequency (or line) spectrum is the plot of A_n and ϕ_n or $|c_n|$ and θ_n versus frequency.
8. The rms value of a periodic function is given by

$$F_{\text{rms}} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2}$$

The power dissipated by a 1-Ω resistance is

$$P_{1\Omega} = F_{\text{rms}}^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=-\infty}^{\infty} |c_n|^2$$

This relationship is known as *Parseval's theorem*.

9. Using *PSpice*, a Fourier analysis of a circuit can be performed in conjunction with the transient analysis.
10. Fourier series find application in spectrum analyzers and filters. The spectrum analyzer is an instrument that displays the discrete Fourier spectra of an input signal, so that an analyst can determine the frequencies and relative energies of the signal's components. Because the Fourier spectra are discrete spectra, filters can be designed for great effectiveness in blocking frequency components of a signal that are outside a desired range.

REVIEW QUESTIONS

16.1 Which of the following cannot be a Fourier series?

(a) $t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5}$
(b) $5\sin t + 3\sin 2t - 2\sin 3t + \sin 4t$
(c) $\sin t - 2\cos 3t + 4\sin 4t + \cos 4t$
(d) $\sin t + 3\sin 2.7t - \cos\pi t + 2\tan\pi t$
(e) $1 + e^{-j\pi t} + \frac{e^{-j2\pi t}}{2} + \frac{e^{-j3\pi t}}{3}$

16.2 If $f(t) = t, 0 < t < \pi, f(t + n\pi) = f(t)$, the value of ω_0 is

(a) 1 (b) 2 (c) π (d) 2π

16.3 Which of the following are even functions?

(a) $t + t^2$ (b) $t^2\cos t$ (c) e^{t^2}
(d) $t^2 + t^4$ (e) $\sinh t$

16.4 Which of the following are odd functions?

(a) $\sin t + \cos t$ (b) $t\sin t$
(c) $t\ln t$ (d) $t^3\cos t$
(e) $\sinh t$

16.5 If $f(t) = 10 + 8\cos t + 4\cos 3t + 2\cos 5t + \cdots$, the magnitude of the dc component is:

(a) 10 (b) 8 (c) 4
(d) 2 (e) 0

16.6 If $f(t) = 10 + 8\cos t + 4\cos 3t + 2\cos 5t + \cdots$, the angular frequency of the 6th harmonic is

(a) 12 (b) 11 (c) 9
(d) 6 (e) 1

16.7 The function in Fig. 16.14 is half-wave symmetric.

(a) True (b) False

16.8 The plot of $|c_n|$ versus $n\omega_0$ is called:

(a) complex frequency spectrum
(b) complex amplitude spectrum
(c) complex phase spectrum

16.9 When the periodic voltage $2 + 6\sin\omega_0 t$ is applied to a 1-Ω resistor, the integer closest to the power (in watts) dissipated in the resistor is:

(a) 5 (b) 8 (c) 20
(d) 22 (e) 40

16.10 The instrument for displaying the spectrum of a signal is known as:

(a) oscilloscope (b) spectrogram
(c) spectrum analyzer (d) Fourier spectrometer

Answers: 16.1a,d, 16.2b, 16.3b,c,d, 16.4d,e, 16.5a, 16.6d, 16.7a, 16.8b, 16.9d ,16.10c.

PROBLEMS

Section 16.2 Trigonometric Fourier Series

16.1 Evaluate each of the following functions and see if it is periodic. If periodic, find its period.

(a) $f(t) = \cos\pi t + 2\cos 3\pi t + 3\cos 5\pi t$
(b) $y(t) = \sin t + 4\cos 2\pi t$
(c) $g(t) = \sin 3t \cos 4t$
(d) $h(t) = \cos^2 t$
(e) $z(t) = 4.2\sin(0.4\pi t + 10^\circ) + 0.8\sin(0.6\pi t + 50^\circ)$
(f) $p(t) = 10$
(g) $q(t) = e^{-\pi t}$

16.2 Determine the period of these periodic functions:

(a) $f_1(t) = 4\sin 5t + 3\sin 6t$

(b) $f_2(t) = 12 + 5\cos 2t + 2\cos(4t + 45^\circ)$

(c) $f_3(t) = 4\sin^2 600\pi t$

(d) $f_4(t) = e^{j10t}$

16.3 Give the Fourier coefficients a_0, a_n, and b_n of the waveform in Fig. 16.47. Plot the amplitude and phase spectra.

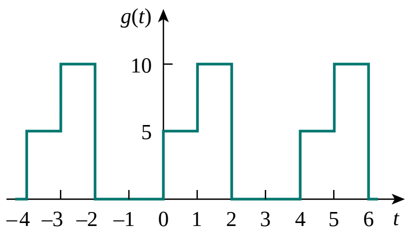


Figure 16.47 For Prob. 16.3.

16.4 Find the Fourier series expansion of the backward sawtooth waveform of Fig. 16.48. Obtain the amplitude and phase spectra.

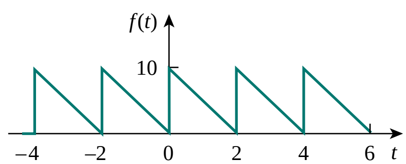


Figure 16.48 For Probs. 16.4 and 16.50.

***16.5** A voltage source has a periodic waveform defined over its period as

$$v(t) = t(2\pi - t)\ \text{V}, \qquad 0 < t < 2\pi$$

Find the Fourier series for this voltage.

16.6 A periodic function is defined over its period as

$$h(t) = \begin{cases} 10\sin t, & 0 < t < \pi \\ 20\sin(t - \pi), & \pi < t < 2\pi \end{cases}$$

Find the Fourier series of $h(t)$.

16.7 Find the quadrature (cosine and sine) form of the Fourier series

$$f(t) = 2 + \sum_{n=1}^{\infty} \frac{10}{n^3 + 1}\cos\left(2nt + \frac{n\pi}{4}\right)$$

16.8 Express the Fourier series

$$f(t) = 10 + \sum_{n=1}^{\infty} \frac{4}{n^2 + 1}\cos 10nt + \frac{1}{n^3}\sin 10nt$$

(a) in a cosine and angle form,

(b) in a sine and angle form.

16.9 The waveform in Fig. 16.49(a) has the following Fourier series:

$$v_1(t) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos\pi t + \frac{1}{9}\cos 3\pi t + \frac{1}{25}\cos 5\pi t + \cdots\right)\ \text{V}$$

Obtain the Fourier series of $v_2(t)$ in Fig. 16.49(b).

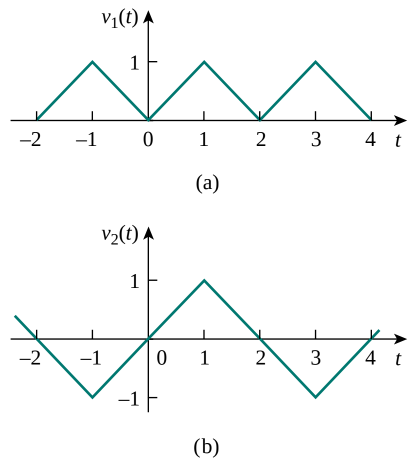


Figure 16.49 For Probs. 16.9 and 16.52.

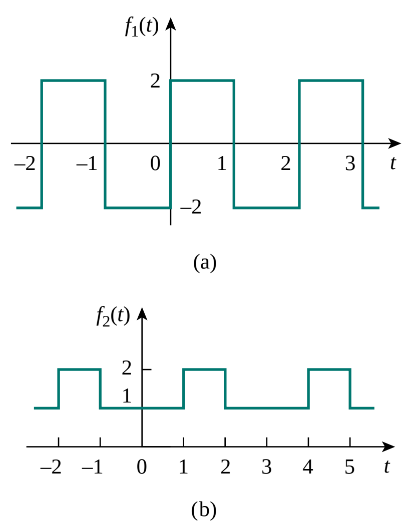
Section 16.3 Symmetry Considerations

16.10 Determine if these functions are even, odd, or neither.

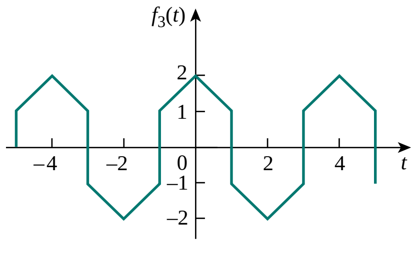
(a) $1 + t$ (b) $t^2 - 1$ (c) $\cos n\pi t \sin n\pi t$

(d) $\sin^2 \pi t$ (e) e^{-t}

16.11 Determine the fundamental frequency and specify the type of symmetry present in the functions in Fig. 16.50.



*An asterisk indicates a challenging problem.



(c)

Figure 16.50 For Probs. 16.11 and 16.48.

16.12 Obtain the Fourier series expansion of the function in Fig. 16.51.

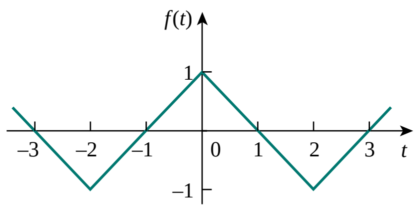


Figure 16.51 For Prob. 16.12.

16.13 Find the Fourier series for the signal in Fig. 16.52. Evaluate $f(t)$ at $t = 2$ using the first three nonzero harmonics.

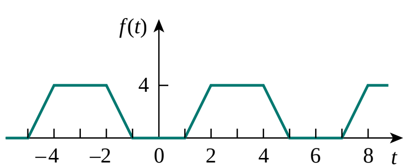


Figure 16.52 For Probs. 16.13 and 16.51.

16.14 Determine the trigonometric Fourier series of the signal in Fig. 16.53.

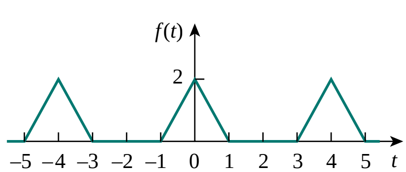


Figure 16.53 For Prob. 16.14.

16.15 Calculate the Fourier coefficients for the function in Fig. 16.54.

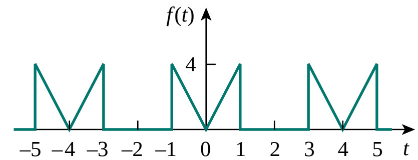


Figure 16.54 For Prob. 16.15.

16.16 Find the Fourier series of the function shown in Fig. 16.55.

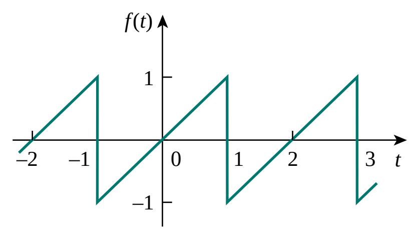


Figure 16.55 For Prob. 16.16.

16.17 In the periodic function of Fig. 16.56,

(a) find the trigonometric Fourier series coefficients a_2 and b_2,

(b) calculate the magnitude and phase of the component of $f(t)$ that has $\omega_n = 10$ rad/s,

(c) use the first four nonzero terms to estimate $f(\pi/2)$,

(d) show that

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

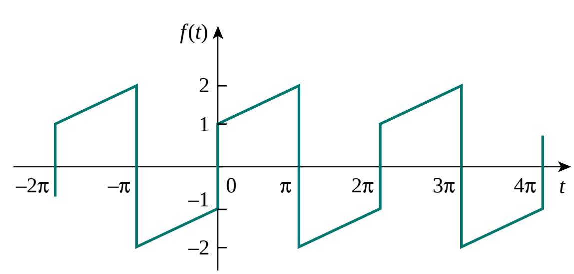


Figure 16.56 For Prob. 16.17.

16.18 Determine the Fourier series representation of the function in Fig. 16.57.

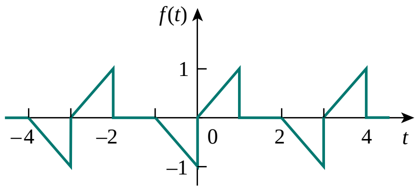


Figure 16.57 For Prob. 16.18.

16.19 Find the Fourier series representation of the signal shown in Fig. 16.58.

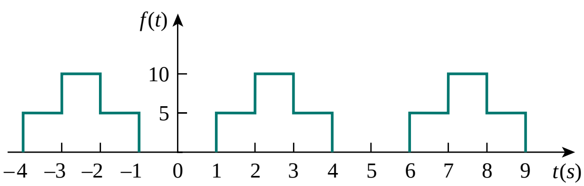


Figure 16.58 For Prob. 16.19.

16.20 For the waveform shown in Fig. 16.59 below,

(a) specify the type of symmetry it has,

(b) calculate a_3 and b_3,

(c) find the rms value using the first five nonzero harmonics.

16.21 Obtain the trigonometric Fourier series for the voltage waveform shown in Fig. 16.60.

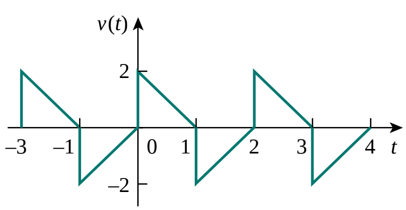


Figure 16.60 For Prob. 16.21.

16.22 Determine the Fourier series expansion of the sawtooth function in Fig. 16.61.

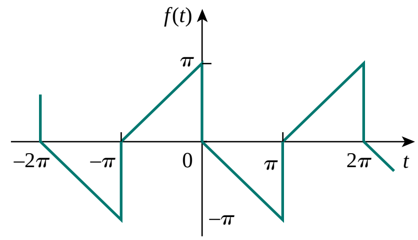


Figure 16.61 For Prob. 16.22.

Section 16.4 Circuit Applications

16.23 Find $i(t)$ in the circuit of Fig. 16.62 given that

$$i_s(t) = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos 3nt \text{ A}$$

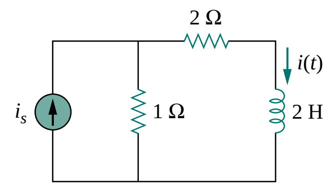


Figure 16.62 For Prob. 16.23.

16.24 Obtain $v_o(t)$ in the network of Fig. 16.63 if

$$v(t) = \sum_{n=1}^{\infty} \frac{10}{n^2} \cos\left(nt + \frac{n\pi}{4}\right) \text{ V}$$

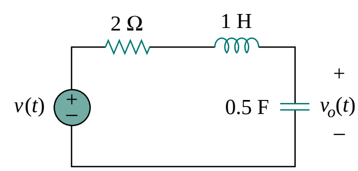


Figure 16.63 For Prob. 16.24.

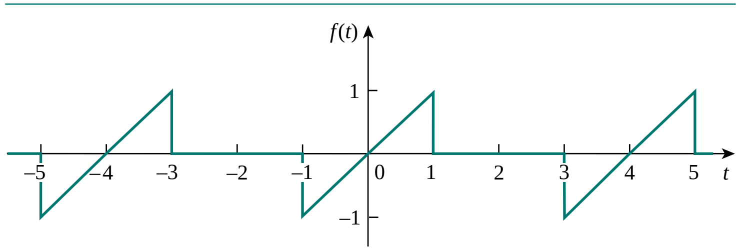


Figure 16.59 For Prob. 16.20.

16.25 If v_s in the circuit of Fig. 16.64 is the same as function $f_2(t)$ in Fig. 16.50(b), determine the dc component and the first three nonzero harmonics of $v_o(t)$.

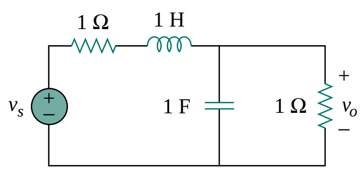


Figure 16.64 For Prob. 16.25.

16.26 Determine $i_o(t)$ in the circuit of Fig. 16.65 if

$$v_s(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \left(\frac{-1}{n\pi} \sin \frac{n\pi}{2} \cos nt + \frac{3}{n\pi} \sin nt \right)$$

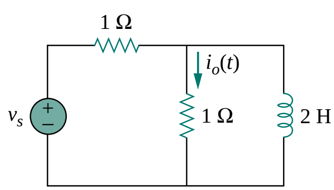


Figure 16.65 For Prob. 16.26.

16.27 The periodic voltage waveform in Fig. 16.66(a) is applied to the circuit in Fig. 16.66(b). Find the voltage $v_o(t)$ across the capacitor.

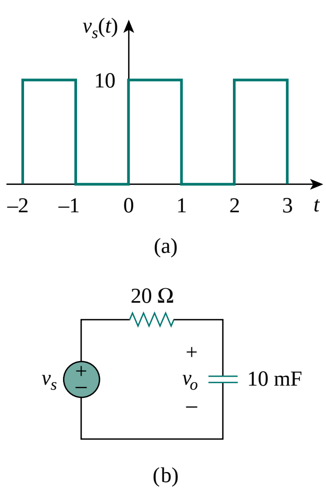


Figure 16.66 For Prob. 16.27.

16.28 If the periodic voltage in Fig. 16.67(a) is applied to the circuit in Fig. 16.67(b), find $i_o(t)$.

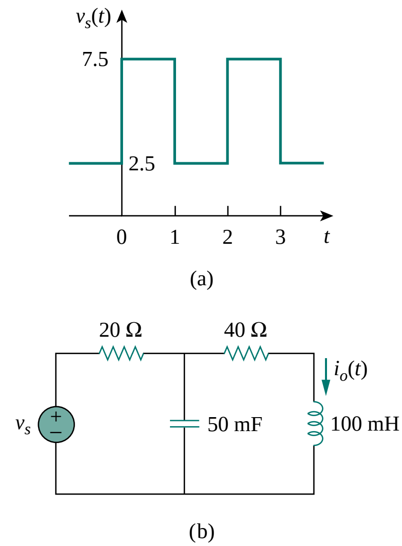


Figure 16.67 For Prob. 16.28.

***16.29** The signal in Fig. 16.68(a) is applied to the circuit in Fig. 16.68(b). Find $v_o(t)$.

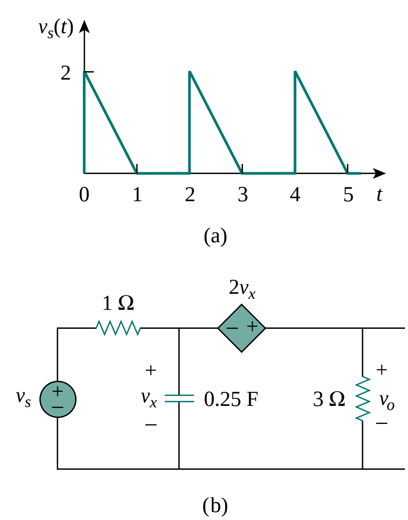


Figure 16.68 For Prob. 16.29.

16.30 The full-wave rectified sinusoidal voltage in Fig. 16.69(a) is applied to the lowpass filter in Fig. 16.69(b). Obtain the output voltage $v_o(t)$ of the filter.

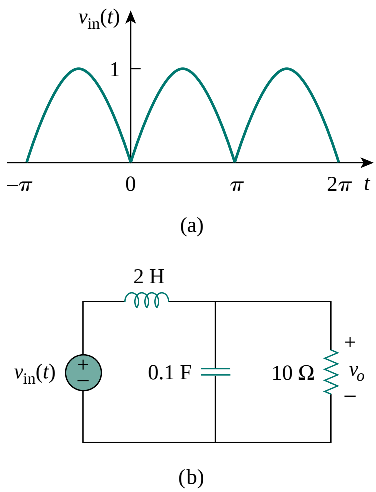


Figure 16.69 For Prob. 16.30.

Section 16.5 Average Power and RMS Values

16.31 The voltage across the terminals of a circuit is

$$v(t) = 30 + 20\cos(60\pi t + 45^\circ) + 10\cos(60\pi t - 45^\circ)\text{ V}$$

If the current entering the terminal at higher potential is

$$i(t) = 6 + 4\cos(60\pi t + 10^\circ) - 2\cos(120\pi t - 60^\circ)\text{ A}$$

find:

(a) the rms value of the voltage,

(b) the rms value of the current,

(c) the average power absorbed by the circuit.

16.32 A series RLC circuit has $R = 10\ \Omega$, $L = 2$ mH, and $C = 40\ \mu$F. Determine the effective current and average power absorbed when the applied voltage is

$$v(t) = 100\cos 1000t + 50\cos 2000t + 25\cos 3000t\text{ V}$$

16.33 Consider the periodic signal in Fig. 16.53. (a) Find the actual rms value of $f(t)$. (b) Use the first five nonzero harmonics of the Fourier series to obtain an estimate for the rms value.

16.34 Calculate the average power dissipated by the 10-Ω resistor in the circuit of Fig. 16.70 if

$$i_s(t) = 3 + 2\cos(50t - 60^\circ) + 0.5\cos(100t - 120^\circ)\text{ A}$$

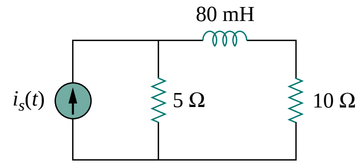


Figure 16.70 For Prob. 16.34.

16.35 For the circuit in Fig. 16.71,

$$i(t) = 20 + 16\cos(10t + 45^\circ) + 12\cos(20t - 60^\circ)\text{ mA}$$

(a) find $v(t)$, and

(b) calculate the average power dissipated in the resistor.

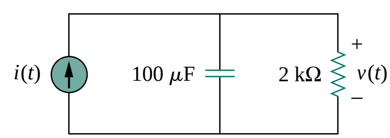


Figure 16.71 For Prob. 16.35.

Section 16.6 Exponential Fourier Series

16.36 Obtain the exponential Fourier series for $f(t) = t$, $-1 < t < 1$, with $f(t + 2n) = f(t)$.

16.37 Determine the exponential Fourier series for $f(t) = t^2$, $-\pi < t < \pi$, with $f(t + 2\pi n) = f(t)$.

16.38 Calculate the complex Fourier series for $f(t) = e^t$, $-\pi < t < \pi$, with $f(t + 2\pi n) = f(t)$.

16.39 Find the complex Fourier series for $f(t) = e^{-t}$, $0 < t < 1$, with $f(t + n) = f(t)$.

16.40 Find the exponential Fourier series for the function in Fig. 16.72.

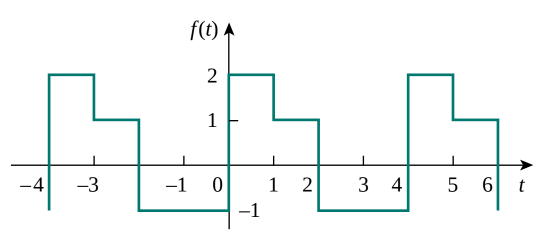


Figure 16.72 For Prob. 16.40.

16.41 Obtain the exponential Fourier series expansion of the half-wave rectified sinusoidal current of Fig. 16.73.

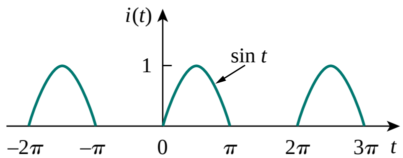


Figure 16.73 For Prob. 16.41.

16.42 The Fourier series trigonometric representation of a periodic function is

$$f(t) = 10 + \sum_{n=1}^{\infty}\left(\frac{1}{n^2+1}\cos n\pi t + \frac{n}{n^2+1}\sin n\pi t\right)$$

Find the exponential Fourier series representation of $f(t)$.

16.43 The coefficients of the trigonometric Fourier series representation of a function are:

$$b_n = 0, \qquad a_n = \frac{6}{n^3-2}, \qquad n = 0, 1, 2, \ldots$$

If $\omega_n = 50n$, find the exponential Fourier series for the function.

16.44 Find the exponential Fourier series of a function which has the following trigonometric Fourier series coefficients

$$a_0 = \frac{\pi}{4}, \qquad b_n = \frac{(-1)^n}{n}, \qquad a_n = \frac{(-1)^n - 1}{\pi n^2}$$

Take $T = 2\pi$.

16.45 The complex Fourier series of the function in Fig. 16.74(a) is

$$f(t) = \frac{1}{2} - \sum_{n=-\infty}^{\infty}\frac{je^{-j(2n+1)t}}{(2n+1)\pi}$$

Find the complex Fourier series of the function $h(t)$ in Fig. 16.74(b).

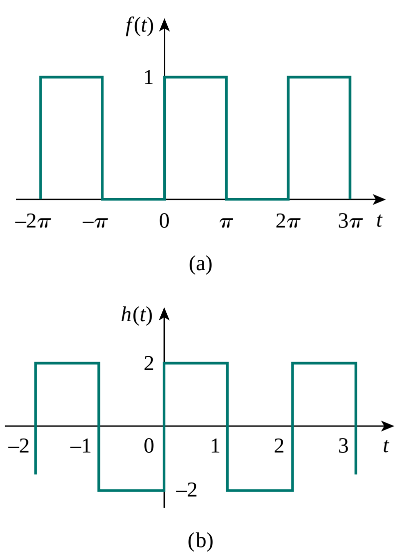


Figure 16.74 For Prob. 16.45.

16.46 Obtain the complex Fourier coefficients of the signal in Fig. 16.56.

16.47 The spectra of the Fourier series of a function are shown in Fig. 16.75. (a) Obtain the trigonometric Fourier series. (b) Calculate the rms value of the function.

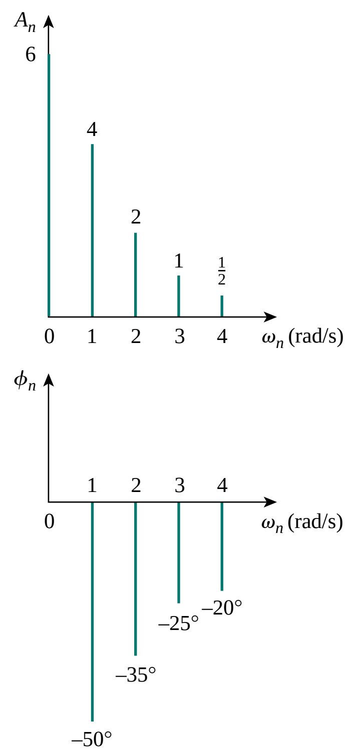


Figure 16.75 For Prob. 16.47.

16.48 Plot the amplitude spectrum for the signal $f_2(t)$ in Fig. 16.50(b). Consider the first five terms.

16.49 Given that

$$f(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty}\left(\frac{20}{n^2\pi^2}\cos 2nt - \frac{3}{n\pi}\sin 2nt\right)$$

plot the first five terms of the amplitude and phase spectra for the function.

Section 16.7 Fourier Analysis with *PSpice*

16.50 Determine the Fourier coefficients for the waveform in Fig. 16.48 using *PSpice*.

16.51 Calculate the Fourier coefficients of the signal in Fig. 16.52 using *PSpice*.

16.52 Use *PSpice* to obtain the Fourier coefficients of the waveform in Fig. 16.49(a).

16.53 Rework Prob. 16.29 using *PSpice*.

16.54 Use *PSpice* to solve Prob. 16.28.

Section 16.8 Applications

16.55 The signal displayed by a medical device can be approximated by the waveform shown in Fig. 16.76. Find the Fourier series representation of the signal.

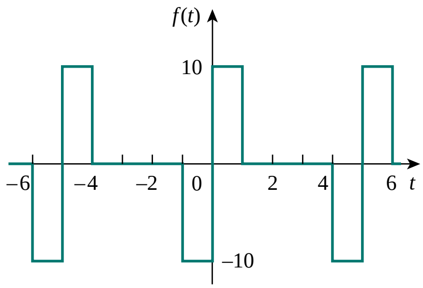


Figure 16.76 For Prob. 16.55.

16.56 A spectrum analyzer indicates that a signal is made up of three components only: 640 kHz at 2 V, 644 kHz at 1 V, 636 kHz at 1 V. If the signal is applied across a 10-Ω resistor, what is the average power absorbed by the resistor?

16.57 A certain band-limited periodic current has only three frequencies in its Fourier series representation: dc, 50 Hz, and 100 Hz. The current may be represented as

$$i(t) = 4 + 6\sin 100\pi t + 8\cos 100\pi t - 3\sin 200\pi t - 4\cos 200\pi t \text{ A}$$

(a) Express $i(t)$ in amplitude-phase form.

(b) If $i(t)$ flows through a 2-Ω resistor, how many watts of average power will be dissipated?

16.58 The signal in Fig. 16.66(a) is applied to the high-pass filter in Fig. 16.77. Determine the value of R such that the output signal $v_o(t)$ has an average power of least 70 percent of the average power of the input signal.

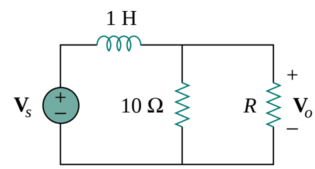


Figure 16.77 For Prob. 16.58.

COMPREHENSIVE PROBLEMS

16.59 The voltage across a device is given by

$$v(t) = -2 + 10\cos 4t + 8\cos 6t + 6\cos 8t - 5\sin 4t - 3\sin 6t - \sin 8t \text{ V}$$

Find:

(a) the period of $v(t)$,

(b) the average value of $v(t)$,

(c) the effective value of $v(t)$.

16.60 A certain band-limited periodic voltage has only three harmonics in its Fourier series representation. The harmonics have the following rms values: fundamental 40 V, third harmonic 20 V, fifth harmonic 10 V.

(a) If the voltage is applied across a 5-Ω resistor, find the average power dissipated by the resistor.

(b) If a dc component is added to the periodic voltage and the measured power dissipated increases by 5 percent, determine the value of the dc component added.

16.61 Write a program to compute the Fourier coefficients (up to the 10th harmonic) of the square wave in Table 16.3 with $A = 10$ and $T = 2$.

16.62 Write a computer program to calculate the exponential Fourier series of the half-wave rectified sinusoidal current of Fig. 16.73. Consider terms up to the 10th harmonic.

16.63 Consider the full-wave rectified sinusoidal current in Table 16.3. Assume that the current is passed through a 1-Ω resistor.

(a) Find the average power absorbed by the resistor.

(b) Obtain c_n for $n = 1, 2, 3,$ and 4.

(c) What fraction of the total power is carried by the dc component?

(d) What fraction of the total power is carried by the second harmonic ($n = 2$)?

16.64 A band-limited voltage signal is found to have the complex Fourier coefficients presented in the table below. Calculate the average power that the signal would supply a 4-Ω resistor.

$n\omega_0$	$\lvert c_n \rvert$	θ_n
0	10.0	0°
ω	8.5	15°
2ω	4.2	30°
3ω	2.1	45°
4ω	0.5	60°
5ω	0.2	75°